Equivariant birational geometry

*Joint work with* Hassett, Kontsevich, Kresch, Pestun, Yang, Zhang
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When the extension is finite, obtained by adjoining a root of a polynomial $f \in k[x]$, this is the content of Galois theory.
Kummer theory

Assume that the ground field $k$ contains $\zeta_p = \exp(2\pi i/p)$. Then every Galois extension $K/k$ with

$$\text{Gal}(K/k) \cong \mathbb{Z}/p$$

is given by

$$K = k(\sqrt[p]{\kappa}), \quad \kappa \in k,$$

i.e., all such extensions are parametrized by

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$$\sqrt[p]{\kappa} \mapsto \zeta_p^i \sqrt[p]{\kappa}.$$

**Proof:** via **Hilbert Theorem 90**

$$H^1(\text{Gal}(K/k), K^\times) = 0.$$
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Rationality

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These are of particular interest. **Why?** They admit many interesting automorphisms:

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$$(R) \Rightarrow (SR)$$
Equivariant geometry

Let $X$ be a smooth projective algebraic variety over $k$, equipped with a generically free action of a finite group $G$. The action is

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  \item[(SL)] stably linear: if $X \times \mathbb{P}^n$ is linear, for some $n$, with trivial $G$-action on the second factor.
\end{itemize}

$(L) \Rightarrow (SL)$
Nonclosed fields and equivariant geometry

There are intriguing connections between arithmetic geometry, i.e., geometry over nonclosed fields and $G$-equivariant geometry over algebraically closed fields.
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This talk will highlight some of these connections.
Let $k$ be a nonclosed field, with absolute Galois group $\Gamma_k$, and $X$ a smooth projective variety over $k$ with function field $K := k(X)$. Then $\Gamma_k$ will act on $\bar{X}$ and on

- special loci,
- geometric Picard group $\text{Pic}(\bar{X})$ and Brauer group $\text{Br}(\bar{X})$, ...
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- Obstruction to stable rationality:

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for some $k'/k$. 

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- Obstruction to stable rationality:
  \[ H^1(\Gamma_{k'}, \text{Pic}(\bar{X})) \neq 0, \quad \text{for some } k'/k. \]

- Obstruction to stable linearizability:
  \[ H^1(G', \text{Pic}(X)) \neq 0, \quad \text{for some } G' \subseteq G. \]
If $X$ is a (stably) rational variety over a nonclosed field $k$, then $X(k) \neq \emptyset$. 
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If $X$ is a $G$-variety, with linearizable $G$-action, then the $G$-fixed locus $X^G$ could be **empty**.
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There is only one $\mathbb{P}^n$ over every field $k$. There exist (many) nonbirational linear $G$-actions!
A word on cohomology

The computation of

$$H^1(\Gamma_k, \text{Pic}(\overline{X}))$$

requires an explicit presentation of $\text{Pic}(\overline{X})$ as a Galois-module.
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requires an explicit presentation of $\text{Pic}(\bar{X})$ as a Galois-module.

In the $G$-context (over an algebraically closed field), this is easier:

**Bogomolov-Prokhorov (2013), Shinder (2017)**

Let $G$ be cyclic (of order $p$), acting (regularly) on a smooth rational surface $X$. If $X^G$ contains a curve of genus $g \geq 1$ then

$$H^1(G, \text{Pic}(X)) = (\mathbb{Z}/p\mathbb{Z})^{2g}.$$
Recall, for schemes (or stacks),

\[ \text{Br}(X) := H^2(X, \mathbb{G}_m). \]
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If \( X \) is smooth projective and rational over \( k \) then

\[ \text{Br}(X) = \text{Br}(k). \]
Let $X$ be a rational $G$-surface over $k = \bar{k}$. There is an exact sequence:

$$0 \to \text{Hom}(G, k^\times) \to \text{Pic}(X, G) \to \text{Pic}(X)^G \xrightarrow{\delta_2(G)} H^2(G, k^\times) \to \text{Br}([X/G]) \to H^1(G, \text{Pic}(X)) \xrightarrow{\delta_3(G)} H^3(G, k^\times).$$
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Note that:

- Both $\delta_2$ and $\delta_3$ are zero, provided $G$ has a fixed point on $X$.
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$$\rightarrow \text{Br}([X/G]) \rightarrow H^1(G, \text{Pic}(X)) \xrightarrow{\delta_3(G)} H^3(G, k^\times).$$

Note that:

- Both $\delta_2$ and $\delta_3$ are zero, provided $G$ has a fixed point on $X$.
- If $G$ is cyclic, then $H^2(G, k^\times) = 0$. 
• The Amitsur group $\text{Am}(\chi, H) = \text{Im}(\delta_2(H)), \quad H \subseteq G,$
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Let $X$ be a rational surface, over $k = \bar{k}$, of characteristic zero. We produce

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Let $X$ be a rational surface, over $k = \bar{k}$, of characteristic zero. We produce

- a recipe to compute
  \[ \text{Br}([X/G]), \]
- examples of $G$-actions with nontrivial
  \[ \text{Br}([X/G]), \quad H^1(G, \text{Pic}(X)), \quad \delta_3(G). \]
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In particular,

\[ \text{Cr}_1(k) = \text{Aut}(k(x)/k) = \text{Aut}(\mathbb{P}^1_k) = \text{PGL}_2(k). \]
Curves

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Finite subgroups of \( \text{Cr}_1 \):

- cyclic \( C_n \)
- dihedral \( D_{2n} \)
- \( A_4 \), \( S_4 \), \( A_5 \).
What about arbitrary $k$?
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In characteristic zero: $Cr_1(k)$ contains

- $C_n, D_{2n}$ iff $k$ contains $\zeta_n + \zeta_n^{-1}$,
What about arbitrary $k$?

In characteristic zero: $C_{r_1}(k)$ contains

- $C_n, D_{2n}$ iff $k$ contains $\zeta_n + \zeta_n^{-1}$,
- $A_4$ and $S_4$ iff $-1$ is a sum of two squares in $k$,
Finite subgroups of $Cr_1(k)$

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- $C_n, D_{2n}$ iff $k$ contains $\zeta_n + \zeta_n^{-1}$,
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- $A_5$ iff $-1$ is a sum of two squares and $5$ is a square in $k$. 
Finite subgroups of $\text{Cr}_1(k)$

What about positive characteristic?
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- it \( k \) is separably closed then two isomorphic subgroups are conjugated,
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- similar description for $D_n$.  

Finite subgroups of $C_{r_1}(k)$
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- \( C_2 \) are parametrized by
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- similar description for \( D_n \),
- \( C_2 \oplus C_2 \) are parametrized by subgroups of order 4 in \( k^\times / (k^\times)^2 \), such that for every pair of generators \( a, b \) one has
  \[
  (-a, -b)_2 = 0 \in Br(k), \quad (\text{Hilbert symbol}).
  \]
Over algebraically closed $k$,

\[ \text{rationality} \Leftrightarrow \text{stable rationality}. \]

This equivalence can fail over nonclosed fields $k$.

**Approach via classification:** Del Pezzo surfaces, conic bundles, ...
Over nonclosed fields $k$, the **rationality problem** is also settled:

A geometrically rational surface $X$ over $k$ is rational iff

- $X(k) \neq \emptyset$ and
- $\text{Pic}(\bar{X})$ is a $\Gamma_k$-permutation module, generated by (orbits of) classes of exceptional curves.
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In particular,

$$H^1(\Gamma_{k'}, \text{Pic}(\tilde{X})) = 0,$$

for all $k'/k$. 
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Del Pezzo surfaces of degree $\geq 5$ with $X(k) \neq \emptyset$ are rational.
The **stable rationality problem** is still open.
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### Conjecture

A geometrically rational surface $X$ over $k$ is stably rational iff

- $X(k) \neq \emptyset$ and
- $\text{Pic}(\bar{X})$ is a stably permutation module.
Let $X$ be a DP4 with

$$H^1(\Gamma_{k'}, \text{Pic}(\bar{X})) = 0, \text{ for all } k'/k.$$ 

Then $X$ is one of the following

- $I_0$: $x^2 - ay^2 = f_3(s)$, with $\text{disc}(f_3) = a$.
- $I_1$: e.g., $x^2 - sy^2 = (s - 3)(s + 3)(s^3 + 9)$.
- $I_2$: e.g., $x^2 - sy^2 = f_2(s) \cdot f_3(s) + \text{conditions}$.
- $I_3$: e.g., $x^2 - sy^2 = -(s^2 - 3)(s^3 + 3)$. 


If $X$ is of type $I_0$, with $X(k) \neq \emptyset$. Then $X$ is stably rational over $k$. 

Kunyavski, Skorobogatov, Tsfasman (1989)
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Let $X$ be a DP4 with

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If $X$ is of type $l_0$, with $X(k) \neq \emptyset$. Then $X$ is stably rational over $k$. 
Identify subgroups of $W(E_6), W(E_7), W(E_8)$ giving rise to

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Let $X$ be a minimal such Del Pezzo surface. Then $d \leq 2$, and

- **deg = 2**: 3 groups ($S_3^2$, $C_2 \times S_4$, $S_5$, and their subgroups), 14 types of conic bundles; 3 groups (4 types) of nonconic examples
- **deg = 1**: 3 groups (10 types) of conic bundles

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- **deg = 1**: 3 groups (10 types) of conic bundles

There are no cyclic groups on the list! Moreover,

**Corollary (strengthening of Segre)**

A minimal cubic surface is not stably rational.
Let $X$ be a $G$-DP surface of degree $\leq 4$ such that

- $H^1(G', \text{Pic}(X)) = 0$, for all $G' \subseteq G$,
- $X \not\sim_G \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. 

Conjecture: This action is stably linearizable.
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Then $X$ is $G$-birational to a complete intersection of two quadrics in $\mathbb{P}^4$,

$$x_1^2 + \zeta x_2^2 + \zeta^2 x_3^2 + x_4^2 = x_1^2 + \zeta^2 x_2^2 + \zeta x_3^2 + x_5^2 = 0,$$

with $\zeta = \zeta_3$, and an action of $G = \mathbb{Z}/3 \rtimes \mathbb{Z}/4$ generated by

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_1, \zeta x_4, \zeta^2 x_5),$$

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**Conjecture**

This action is stably linearizable.
In contrast to the situation over nonclosed fields, interesting things happen in degree $\geq 5$!
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Study stable linearizability of $G$-DP surfaces of degree $\geq 5$:

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Study stable linearizability of $G$-DP surfaces of degree $\geq 5$:

- there exist nonlinear but stably linear actions in degrees 8, 6, 5,
- complete answer for quadrics.
Assume that \( k = \bar{k} \), and consider \( \text{Aut}(k(\mathbb{P}^2)/k) \):

- **linear**, i.e., elements of \( \text{PGL}_3 \), defined on all points of \( \mathbb{P}^2 \); we have

  \[
  \text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(k).
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  \[ \text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(k). \]

- **Cremona involution**:

  \[(x : y : z) \mapsto \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right). \]
Birational automorphisms of $\mathbb{P}^2$

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  $$\text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(k).$$

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  $$\begin{pmatrix} x : y : z \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \end{pmatrix}.$$  

The **Cremona group**

$$\text{BirAut}(\mathbb{P}^2_k) = \text{Aut}(k(\mathbb{P}^2)/k) = \text{Cr}_2(k),$$

is generated by the maps above.
Solved problem
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### Open problem
What are the finite subgroups of the $\text{Cr}_2$, up to conjugation?
There is an enormous literature concerning $\text{Cr}_2$. 
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Just a few references:

- Bertini, Castelnuovo, Kantor, . . .
- Beauville, de Fernex (2004) – cyclic subgroups
The plane Cremona group

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- Dolgachev-Iskovskikh (2006) – classification of finite subgroups,
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- Bertini, Castelnuovo, Kantor, . . .
- Beauville, de Fernex (2004) – cyclic subgroups
- Dolgachev-Iskovskikh (2006) – classification of finite subgroups, , with follow-up work by Prokhorov, Beauville, Ch. Xu, Blanc, ...
The plane Cremona group

There is an enormous literature concerning $\text{Cr}_2$.

Just a few references:

- Bertini, Castelnuovo, Kantor, . . .
- Beauville, de Fernex (2004) – cyclic subgroups
- Dolgachev-Duncan (2014) – classification of groups admitting actions with a fixed point, on some model
• \( R \): Yasinsky (2015) – classification of subgroups of odd order
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• \textbf{R}: Lamy-Zimmerman (2017) – nontrivial quotients to (many) $\mathbb{Z}/2\mathbb{Z}$
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• $\mathbb{R}$: Cheltsov-Mangolte-Yasinsky-Zimmerman (2022) – classification of involutions
Cremona group over nonclosed fields

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- $\mathbb{R}$: Cheltsov-Mangolte-Yasinsky-Zimmerman (2022) – classification of involutions
- $\mathbb{F}_q$: Genevois, Lonjou, Urech (2021) – connection to cryptography
Usnich/Kontsevich (2008)

Connection to deformation quantization, cluster algebras; study the subgroup

\[ \text{Symp}_2(k) \subset \text{Cr}_2(k) \]

preserving the Poisson bracket \( \{x, y\} := xy \).
Proposition 8.2.1: Automorphisms of Cubic Surfaces

Let $G$ be an abelian group of automorphisms of a non-singular cubic surface $S = V(F)$, such that $\text{rk Pic}(S)^G = 1$. Then, up to isomorphism, we are in one of the following cases:

<table>
<thead>
<tr>
<th>name of $(G, S)$</th>
<th>structure of $G$</th>
<th>generators of $G$</th>
<th>equation of $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td>$\mathbb{Z}/3\mathbb{Z}$</td>
<td>$[\omega : 1 : 1 : 1]$</td>
<td>$w^3 + L_3(x, y, z)$</td>
</tr>
<tr>
<td>3.6.1</td>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td>$[\omega : 1 : 1 : -1]$</td>
<td>$w^3 + x^3 + y^3 + xz^2 + \lambda yz^2$</td>
</tr>
<tr>
<td>3.33.1</td>
<td>$(\mathbb{Z}/3\mathbb{Z})^2$</td>
<td>$[\omega : 1 : 1 : 1], [1 : 1 : 1 : \omega]$</td>
<td>$w^3 + x^3 + y^3 + z^3$</td>
</tr>
<tr>
<td>3.9</td>
<td>$\mathbb{Z}/9\mathbb{Z}$</td>
<td>$[\zeta_9 : 1 : \omega : \omega^2]$</td>
<td>$w^3 + xz^2 + x^2y + y^2z$</td>
</tr>
<tr>
<td>3.33.2</td>
<td>$(\mathbb{Z}/3\mathbb{Z})^2$</td>
<td>$[\omega : 1 : 1 : 1], [1 : 1 : \omega : \omega^2]$</td>
<td>$w^3 + x^3 + y^3 + z^3 + \lambda xyz$</td>
</tr>
<tr>
<td>3.12</td>
<td>$\mathbb{Z}/12\mathbb{Z}$</td>
<td>$[\omega : 1 : -1 : i]$</td>
<td>$w^3 + x^3 + yz^2 + y^2x$</td>
</tr>
<tr>
<td>3.36</td>
<td>$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$</td>
<td>$[\omega : 1 : 1 : 1], [1 : 1 : -1 : \omega]$</td>
<td>$w^3 + x^3 + xy^2 + z^3$</td>
</tr>
<tr>
<td>3.333</td>
<td>$(\mathbb{Z}/3\mathbb{Z})^3$</td>
<td>$[\omega : 1 : 1 : 1], [1 : \omega : 1 : 1]$</td>
<td>$w^3 + x^3 + y^3 + z^3$</td>
</tr>
<tr>
<td>3.6.2</td>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td>$[1 : -1 : \omega : \omega^2]$</td>
<td>$wx^2 + w^3 + y^3 + z^3 + \lambda wxyz$</td>
</tr>
</tbody>
</table>

where $L_3$ denotes a non-singular form of degree 3, $\lambda \in \mathbb{C}$ is a parameter such that the surface is non-singular and $\omega = e^{2\pi i/3}, \zeta_9 = e^{2\pi i/9}$.

Furthermore, all the cases above are minimal pairs $(G, S)$ with $\text{rk Pic}(S)^G = 1$. 
Consider $G := \mathbb{Z}/2 \oplus \mathbb{Z}/4$ generated by

$$(x : y : z) \mapsto (yz : xy : -xz),$$

$$(x : y : z) \mapsto (yz(y - z) : xz(y + z) : xy(y + z)).$$

Then $G \subset C_{r_2}$ is not conjugated to any subgroup of $\text{PGL}_3$ or $\text{PGL}_2 \times \text{PGL}_2$.
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Then $G \subset \text{Cr}_2$ is not conjugated to any subgroup of $\text{PGL}_3$ or $\text{PGL}_2 \times \text{PGL}_2$. This action is not stably linearizable!
Blanc (2006)

Let $G \subset \text{Cr}_2$ be a finite abelian subgroup.
Blanc (2006)

Let $G \subset \text{Cr}_2$ be a finite abelian subgroup. Assume that no element in $G$ fixes a curve of positive genus, in particular

$$H^1(G', \text{Pic}(X)) = 0,$$

for all $G' \subseteq G$. 

An example
Let $G \subset C_{r_2}$ be a finite abelian subgroup. Assume that no element in $G$ fixes a curve of positive genus, in particular

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for all $G' \subseteq G$.

Then either the action is equivariantly conjugated to

$$G \subseteq \text{Aut}(\mathbb{P}^2), \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1),$$

or
Let $G \subseteq \text{Cr}_2$ be a finite abelian subgroup. Assume that no element in $G$ fixes a curve of positive genus, in particular

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Then either the action is equivariantly conjugated to

$$G \subseteq \text{Aut}(\mathbb{P}^2), \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1),$$

or it is the action above (realized as a regular action on a conic bundle).
Basic strategy:

- The actions are realized as regular actions on \textit{minimal} rational surfaces $X$, 
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- By MMP, $X$ is either a Del Pezzo surface or a conic bundle,
Basic strategy:

- The actions are realized as regular actions on \textit{minimal} rational surfaces $X$,
- By MMP, $X$ is either a Del Pezzo surface or a conic bundle,
- If the (anticanonical) degree is small, the action is \textit{rigid}, and visible via the induced action on the Picard group $\text{Pic}(X)$, i.e., through the Weyl group of the associated root lattice.

$\Rightarrow$ long tables.
<table>
<thead>
<tr>
<th>Type</th>
<th>Order</th>
<th>Structure</th>
<th>$F(T_0, T_1, T_2, T_3)$</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>648</td>
<td>$3^3 : S_4$</td>
<td>$T_0^3 + T_1^3 + T_2^3 + T_3^3$</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>120</td>
<td>$S_5$</td>
<td>$T_0^2T_1 + T_0T_2^2 + T_2T_3^2 + T_3T_1^2$</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>108</td>
<td>$H_3(3) : 4$</td>
<td>$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$</td>
<td>$20a^3 + 8a^6 = 1$</td>
</tr>
<tr>
<td>IV</td>
<td>54</td>
<td>$H_3(3) : 2$</td>
<td>$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$</td>
<td>$a - a^4 \neq 0$, $8a^3 \neq -1$, $20a^3 + 8a^6 \neq 1$</td>
</tr>
<tr>
<td>V</td>
<td>24</td>
<td>$S_4$</td>
<td>$T_0^3 + T_0(T_1^2 + T_2^2 + T_3^2)$ $+aT_1T_2T_3$</td>
<td>$9a^3 \neq 8a$, $8a^3 \neq -1$, $a \neq 0$</td>
</tr>
<tr>
<td>VI</td>
<td>12</td>
<td>$S_3 \times 2$</td>
<td>$T_2^3 + T_3^3 + aT_2T_3(T_0 + T_1) + T_0^3 + T_1^3$</td>
<td>$a \neq 0$, $b \neq 0, 1$</td>
</tr>
<tr>
<td>VII</td>
<td>8</td>
<td>8</td>
<td>$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2$</td>
<td>$a \neq 0$</td>
</tr>
<tr>
<td>VIII</td>
<td>6</td>
<td>$S_3$</td>
<td>$T_2^3 + T_3^3 + aT_2T_3(T_0 + bT_1) + T_0^3 + T_1^3$</td>
<td>$a \neq 0, b \neq 0, 1$</td>
</tr>
<tr>
<td>IX</td>
<td>4</td>
<td>4</td>
<td>$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2 + aT_1^3$</td>
<td>$a \neq 0$</td>
</tr>
<tr>
<td>X</td>
<td>4</td>
<td>$2^2$</td>
<td>$T_0^2(T_1 + T_2 + aT_3) + T_1^3 + T_2^3$ $+T_3^3 + 6bT_1T_2T_3$</td>
<td>$8b^3 \neq -1$</td>
</tr>
<tr>
<td>XI</td>
<td>2</td>
<td>2</td>
<td>$T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$ $+T_0^2(T_1 + bT_2 + cT_3)$</td>
<td>$b^3, c^3 \neq 1$, $b^3 \neq c^3$, $8a^3 \neq -1$,</td>
</tr>
</tbody>
</table>

Table 4. Groups of automorphisms of cubic surfaces.
We do not know whether any two isomorphic non-conjugate subgroups of PGL(3) are conjugate in Cr(2).

9. What is left?

Here we list some problems which have not been yet resolved.

- Find the conjugacy classes in Cr(2) of subgroups of PGL(3). For example, there are two non-conjugate subgroups of PGL(3) isomorphic to $A_5$ and three to $A_6$ which differ by an outer automorphism of the groups. Are they conjugate in Cr(2)?
Problem: How to distinguish equivariant birational types of linear actions?
Problem: How to distinguish equivariant birational types of \textbf{linear} actions? How to distinguish linear actions from nonlinear actions?
**Problem:** How to distinguish equivariant birational types of linear actions? How to distinguish linear actions from nonlinear actions?

**Basic facts:**

- If $X$ is rational and $G$ is *cyclic*, then $X^G \neq \emptyset$. 
Problem: How to distinguish equivariant birational types of linear actions? How to distinguish linear actions from nonlinear actions?

Basic facts:

• If $X$ is rational and $G$ is cyclic, then $X^G \neq \emptyset$.
• If $Y \rightarrow X$ is a $G$-birational map between smooth projective $G$-varieties, and $G$ is abelian, then

$$Y^G \neq \emptyset \iff X^G \neq \emptyset.$$
More precisely, let $X$ be smooth projective of dimension $n$, $G$ abelian, and let $p \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of $G$ in the tangent space to $X$ at $p$. 

Reichstein-Youssin (2002)
More precisely, let $X$ be smooth projective of dimension $n$, $G$ abelian, and let $p \in X^G$. Let $\{a_1, \ldots, a_n\}$ be the characters (weights) of $G$ in the tangent space to $X$ at $p$. Let

$$\det(a_1, \ldots, a_n) = a_1 \wedge \ldots \wedge a_n \in \wedge^n (G^\vee)$$

be the determinant.
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be the determinant.

**Reichstein-Youssin (2002)**

Let $Y \to X$ be a $G$-equivariant blowup. Then $Y$ contains a point $q \in Y^G$ (in the preimage of $p$) with weights $\{b_1, \ldots, b_n\}$ in the tangent space, and such that

$$\det(b_1, \ldots, b_n) = \pm \det(a_1, \ldots, a_n),$$

i.e., this is a **equivariant birational invariant**.
Let $V$ and $W$ be $n$-dimensional faithful representations of an abelian group $G$ of rank $r \leq n$, and

$$a_1, \ldots, a_n, \text{ respectively } b_1, \ldots, b_n,$$

the characters of $G$ appearing in $V$, respectively $W$. Then $V$ and $W$ are $G$-equivariantly birational if and only if

$$a_1 \wedge \cdots \wedge a_n = \pm b_1 \wedge \cdots \wedge b_n.$$

(This condition is meaningful only when $r = n$.)
Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.
• Thus, cyclic linear actions on $\mathbb{P}^n$, with $n \geq 2$, of the same order, are equivariantly birational.

• Note that any two faithful representations of $G$ are equivariantly stably birational.
**First examples: \( \mathbb{P}^2 \)**

Consider an action of \( \mathbb{Z}/p\mathbb{Z} \) on \( X = \mathbb{P}^2 \) given by

\[
(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),
\]

where \( \zeta = \zeta_p, \quad a, b \in \mathbb{Z}/p\mathbb{Z}, \quad \text{gcd}(a, b, p) = 1, \quad a \neq b. \)

Fixed points are

\[
(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).
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Fixed points are

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).$$

Then

$$\beta(X) = [a, b] + [a - b, -b] + [b - a, -a].$$
All such actions are equivalent. Declare $\beta(X) = 0$, i.e.,

$$[a, b] = -[b - a, -a] - [a - b, -b]$$

Allowing

$$[a, b] = -[a, -b]$$

we find

$$[a, b] = [a, b - a] + [a - b, b].$$
both sides of the correspondence (in some sense, quantize the classical mechanical side, while $\beta$-deforming the conformal block side):

$$\Psi(a, \varepsilon_1, \varepsilon_2, m; w, q) = \sum_{n \in \Lambda} \Psi(a + \varepsilon_1 n, \varepsilon_1, \varepsilon_2 - \varepsilon_1, m; w, q) Z(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2, \varepsilon_2, m, q).$$

(1)

Here $\Psi(a, \varepsilon_1, \varepsilon_2, m; w, q)$, $Z(a, \varepsilon_1, \varepsilon_2, m, q)$ are the conformal blocks of the current algebra and $W$-algebra, respectively. The natural habitat for (1) is the four dimensional $\mathcal{N} = 2$ supersymmetric $\Omega$-deformed gauge theory, where it is the relation between the (unnormalized) expectation value $\Psi$ of a surface defect located at the surface $z_2 = 0$ with its own couplings $w$, and the supersymmetric partition function $Z$ of the theory on $\mathbb{R}^4$, with the bulk coupling $q$:

$$q = e^{-\frac{8\pi^2}{g^2}} e^{i\theta}.$$  

(2)

The relation (1) accompanies the well-known equivariant blowup formula

$$Z(a, \varepsilon_1, \varepsilon_2, m, q) = \sum_{n \in \Lambda} Z(a + \varepsilon_1 n, \varepsilon_1, \varepsilon_2 - \varepsilon_1, m, q) Z(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2, \varepsilon_2, m, q).$$

(3)

found in [123].
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found in [123].

Birational types \( B_2(\mathbb{Z}/p\mathbb{Z}) \)

**Generators:** \([a, b], a, b \in \mathbb{Z}/p\mathbb{Z}, \gcd(a, b, p) = 1\)

**Relations:**

- \([a, b] = [b, a]\)
- \([a, b] = [a, b - a] + [a - b, b]\) if \(a \neq b\)
- \([a, a] = [a, 0]\)
Birational types $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

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The $\mathbb{Q}$-rank of $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$ equals

$$\frac{p^2 - 1}{24} + 1.$$
Let $G$ be a finite **abelian** group, and $A = G^\vee$ its group of characters.
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Let $X$ be smooth projective, of dimension $n$, with regular $G$-action. Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of $G$

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

in the tangent space $T_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. 
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$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$
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$$\beta(X) := \sum_{\alpha} [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

Here, we keep **no** information about $F_\alpha$.  

Consider the free abelian group

\[ S_n(G) \]

spanned by unordered tupels

\[ [a_1, \ldots, a_n], \quad a_i \in A, \]
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subject to condition:

\[ (G) \sum_i \mathbb{Z}a_i = A, \]
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$$[a_1, \ldots, a_n], \quad a_i \in A,$$

subject to condition:

$$(G) \sum_i \mathbb{Z}a_i = A,$$

We get a map

$$\{ G\text{-varieties} \} \rightarrow S_n(G)$$

$$X \mapsto \beta(X)$$
Let $Y \to X$ be a $G$-equivariant blowup and impose relations:

$$\beta(Y) - \beta(X) = 0.$$
Birational types $\mathcal{B}_n(G)$

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**All** such relations can be encoded in a compact form:
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$$S_n(G) \to B_n(G),$$

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All such relations can be encoded in a compact form: Consider the quotient

$$S_n(G) \to \mathcal{B}_n(G),$$

by relations

(B) for all $a_1, a_2, b_3, \ldots, b_n \in A$ we have

$$[a_1, a_2, b_3, \ldots b_n] =$$

$$[a_1 - a_2, a_2, b_3, \ldots, b_n] + [a_1, a_2 - a_1, b_3, \ldots, b_n] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_3, \ldots, b_n] \quad \text{ if } a_1 = a_2.$$
The class 

\[ \beta(X) \in \mathcal{B}_n(G) \]

is a well-defined $G$-equivariant birational invariant.
The class

$$\beta(X) \in B_n(G)$$

is a well-defined $G$-equivariant birational invariant.

**Proof:** Equivariant Weak Factorization (Abramovich, Karu, Matsuki, Włodarczyk)
Birational types

For $G = \mathbb{Z}/p\mathbb{Z}$ and $n = 2$, we get $\binom{p}{2}$ linear equations in the same number of variables.
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\[
\text{rk}_Q(B_2(G)) = \frac{p^2 - 1}{24} + 1
\]
Birational types

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$$\text{rk}_\mathbb{Q}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1$$

For $n \geq 3$ the systems of equations are highly overdetermined.
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For $n \geq 3$ the systems of equations are highly overdetermined.

$$\text{rk}_Q(\mathcal{B}_3(G)) = \frac{(p - 5)(p - 7)}{24} = \frac{p^2 - 1}{24} + 1 - \frac{p - 1}{2}$$
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For $n \geq 3$ the systems of equations are highly overdetermined.

$$\text{rk}_\mathbb{Q}(\mathcal{B}_3(G)) \equiv \frac{(p - 5)(p - 7)}{24} = \frac{p^2 - 1}{24} + 1 - \frac{p - 1}{2}$$

Jumps at

$$p = 43, 59, 67, 83, \ldots$$
Birational types

For \( G = \mathbb{Z}/p\mathbb{Z} \) and \( n = 2 \), we get \( \binom{p}{2} \) linear equations in the same number of variables.

\[
\text{rk}_\mathbb{Q}(\mathcal{B}_2(G)) = \frac{p^2 - 1}{24} + 1
\]

For \( n \geq 3 \) the systems of equations are highly overdetermined.

\[
\text{rk}_\mathbb{Q}(\mathcal{B}_3(G)) \approx \frac{(p - 5)(p - 7)}{24} = \frac{p^2 - 1}{24} + 1 - \frac{p - 1}{2}
\]

Jumps at

\( p = 43, 59, 67, 83, \ldots \)

These are interesting groups!
Variant: introduce the quotient

$$
\mu^- : B_n(G) \rightarrow B_n^-(G)
$$

by an **additional** relation

$$
[a_1, a_2, \ldots, a_n] = -[\overline{a_1}, a_2, \ldots, a_n].
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Variant: introduce the quotient

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The class of \( \mathbb{P}^n, n \geq 2 \), with linear action of \( G := \mathbb{Z}/N\mathbb{Z} \) is

- **torsion** in \( \mathcal{B}_n(G) \) and
- **trivial** in \( \mathcal{B}_n^-(G) \).
\[ B_n^- (G) \otimes \mathbb{Q} \cong H^{n(n-1)/2} (\Gamma(G, n), \text{or}_n) = H_0(\Gamma(G, n), \text{St}_n \otimes \text{or}_n) \]

where

- \( \Gamma(G, n) \subset \text{GL}_n(\mathbb{Z}) \)
  
  is a *congruence subgroup*,

- or is the orientation (the sign of the determinant), and

- \( \text{St}_n \) is the *Steinberg representation*. 
We work over a field $k$ of characteristic zero (with enough roots of 1). Let

$$\text{Burn}_n(G) = \text{Burn}_{n,k}(G)$$

be the $\mathbb{Z}$-module, generated by symbols

$$(H, Y \acts K, \beta),$$

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where

- $H \subseteq G$ is an **abelian** subgroup, $Y \subseteq Z_G(H)/H$,
- $K = k(F)$, with generically free $Y$-action, $\text{trdeg}_k(K) = d \leq n$,
- $\beta = (b_1, \ldots, b_{n-d})$, a sequence, up to order, of **nonzero** elements of $H^\vee$, that generate $H^\vee$. 
The symbols are subject to **conjugation** and **blowup** relations:

\[(C): (H, Y \subseteq K, \beta) = (H', Y' \subseteq K, \beta'), \text{ when} \]

\[H' = gHg^{-1}, \quad Y' = \cdots, \quad \text{with } g \in G,\]

and $\beta$ and $\beta'$ are related by conjugation by $g$. 
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\]

and \(\beta\) and \(\beta'\) are related by conjugation by \(g\).

**B1**: \((H, Y \triangleleft K, \beta) = 0\) when \(b_1 + b_2 = 0\).
Equivariant Burnside group: relations

**B2:** \((H, Y \subset K, \beta) = \Theta_1 + \Theta_2\), where

\[
\Theta_1 = \begin{cases} 
0, & \text{if } b_1 = b_2, \\
(H, Y \subset K, \beta_1) + (H, Y \subset K, \beta_2), & \text{otherwise},
\end{cases}
\]

with

\[
\beta_1 := (b_1, b_2 - b_1, b_3, \ldots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \ldots, b_{n-d}),
\]

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Equivariant Burnside group: relations

(B2): \((H, Y \varsubsetneq K, \beta) = \Theta_1 + \Theta_2\), where

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\beta_1 := (b_1, b_2 - b_1, b_3, \ldots, b_{n-d}), \quad \beta_2 := (b_1 - b_2, b_2, b_3, \ldots, b_{n-d}),
\]

and

\[
\Theta_2 = \begin{cases} 
0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\
(\overline{H}, \overline{Y} \varsubsetneq \overline{K}, \overline{\beta}), & \text{otherwise,}
\end{cases}
\]

with

\[
\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \overline{\beta} := (\overline{b}_2, \overline{b}_3, \ldots, \overline{b}_{n-d}), \quad \overline{b}_i \in \overline{H}^\vee.
\]
**Model case:** Blowing up an isolated point (with abelian stabilizer) on a surface.

It will explain the action of $\bar{Y}$ on $\bar{K} = K(t)$. 
Forget the function field of strata with nontrivial stabilizers:

\[ \mathcal{B}C_n(G) \]

is generated by symbols

\( (H, Y, \beta) \),

subject to relations as in the definition of \( \text{Burn}_n(G) \).
Combinatorial version: properties

• There is a natural homomorphism

\[ \text{Burn}_n(G) \to \mathcal{B}C_n(G). \]
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• This group is computable.

• T.– Kaiqi Yang, Zhijia Zhang (2021)

\[ \mathcal{B}C_n(G) = \bigoplus_{[H,Y]} \left( \mathcal{B}_n(H) / \text{certain } (H, Y)-\text{conjugation} \right), \]

over conjugacy classes of pairs \((H, Y)\).
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over conjugacy classes of pairs \((H, Y)\). For \(G\) abelian, one has

\[ \mathcal{B}C_n(G) = \bigoplus_{G' \subseteq G} \bigoplus_{G'' \subseteq G'} \mathcal{B}_n(G''). \]
Equivariant Burnside group

The class

$$[X \ltimes G] \in \text{Burn}_n(G)$$

of a $G$-variety is computed on a standard model $(X, D)$:

- $X$ is smooth projective, $D$ a normal crossings divisor,
- $G$ acts freely on $U := X \setminus D$,
- for every $g \in G$ and every irreducible component $D$, either $g(D) = D$ or $g(D) \cap D = \emptyset$. 
Equivariant Burnside group

Passing to a standard model $X$, define:

$$[X \lhd G] := \sum_H \sum_F (H, Y \lhd k(F), \beta_F(X)) \in \text{Burn}_n(G),$$

where the sum is over (conjugacy classes of) abelian subgroups $H \subseteq G$, and all $F \subseteq X$ with generic stabilizer $H$. 
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- the generic stabilizer $H$, 

Equivariant Burnside group

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- the induced $Y \subseteq Z_G(H)/H$-action on the function field of the subvariety $F \subseteq X$, with generic stabilizer $H$, 
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The symbols record

- the generic stabilizer $H$,
- the induced $Y \subseteq Z_G(H)/H$-action on the function field of the subvariety $F \subset X$, with generic stabilizer $H$,
- the (generic) eigenvalues of $H$ in the normal bundle along $F$. 
The class

\[ [X \curvearrowright G] \in \text{Burn}_n(G) \]

is a well-defined $G$-equivariant birational invariant.
Kresch–T. (2020)

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**Proof:** Equivariant Weak Factorization.
Simplifications arise when we focus on geometric properties of the function fields of strata.
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$$\text{Burn}_{n}^{\text{inc}}(G) \subset \text{Burn}_{n}(G),$$

generated by **incompressible divisor symbols**, i.e.,

$$s = (H, Y \subseteq K, \beta), \quad \text{trdeg}_k(K) = n - 1,$$

$H$ is a nontrivial cyclic group and $\beta = (b)$, a single character, generating $H^\vee$. 
Burnside groups: incompressibles

Simplifications arise when we focus on geometric properties of the function fields of strata; there is a distinguished subgroup

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$$\mathfrak{s} = (H, Y \subset K, \beta), \quad \text{trdeg}_k(K) = n - 1,$$

$H$ is a nontrivial cyclic group and $\beta = (b)$, a single character, generating $H^\vee$, and such that $\mathfrak{s}$ cannot arise from $\Theta_2$ in relation (B2).
The subgroup

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\[ n = 1 \] Every divisor symbol in incompressible.

\[ n = 2 \] A divisor symbol
\[ (H, Y \subset K, \beta), \quad \beta = (b), \]
is compressible if and only if \( Y \) is cyclic and \( K = k(t) \).
Incompressibles

We have a projection

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Given an embedding

\[ \iota: G \hookrightarrow \text{Cr}_n, \]

we obtain a sum of incompressible divisorial symbols

\[ \text{inc}([\iota]) \in \text{Burn}^{\text{inc}}_n(G). \]
Let $G$ be a cyclic group, acting regularly and generically freely on a smooth rational surface over $k = \mathbb{C}$. Then

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Let $G = \langle g \rangle$ be cyclic of order 4, and $X \subset \mathbb{P}(1, 1, 2, 3)$ be

$$w^2 = z^3 + z(ax^4 + bx^2y^2 + cy^4) + xy(a'x^4 + b'x^2y^2 + c'y^4).$$

For general $a, b, c, a', b', c'$, the embeddings

$$\iota, \iota' : G \rightarrow \text{Aut}(X) \subset \text{Cr}_2,$$

where $g$ acts by scalar multiplication on the coordinates $w, x, y, z$ by

$$[i : 1 : -1 : -1], \quad \text{respectively} \quad [-i : 1 : -1 : -1],$$

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Basic terminology: a (faithful) representations $G \to \text{GL}(V)$ is called:

- **intransitive**: if it is reducible, **transitive** if it is irreducible;
Applications: Linear actions

**Basic terminology:** a (faithful) representation $G \to \text{GL}(V)$ is called:

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Basic terminology: a (faithful) representations $G \rightarrow \text{GL}(V)$ is called:

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- **imprimitive** if it is transitive but contains an intransitive **normal** subgroup $G'$; in this case $G/G'$ permutes the $G'$ representations;
- **primitive** if it is neither intransitive, nor imprimitive.
\[ G \subset k^\times \times \text{GL}_2(k); \]
\( \mathbb{P}^2: \text{intransitive} \)

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finite subgroups of \( \text{GL}_2 \) arise as binary extensions of subgroups of \( \text{PGL}_2 \), which in turn are:

\[ C_n, D_{2n}, A_4, S_4, A_5. \]
Example: Extension of $C_3$ by $(\mathbb{Z}/n\mathbb{Z})^2$, with the action

$$(\zeta_n x_0, x_1, x_2), \quad (x_0, \zeta_n x_1, x_2), \quad (x_2, x_0, x_1),$$

together with a cyclic permutation of the coordinates.
$\mathbb{P}^2$: primitive

- $\mathfrak{A}_5$
- $3^2 : \text{SL}_2(\mathbb{F}_3)$, and two of its subgroups
- $\text{PSL}_2(\mathbb{F}_7)$,
- $\mathfrak{A}_6$
The standard approaches to distinguishing $G$-actions, up to birationality, rely on $H^1(G, \text{Pic}(X))$ or birational rigidity.
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---

Theorem (Sakovics, 2019) Let $G = \text{PGL}_3$ be a finite group. Then $\mathbb{P}^2$ is $G$-birationally rigid if and only if $G$ is transitive and not isomorphic to $A_4$ or $S_4$. There are two actions of $A_5$; they are not conjugated in $\text{PGL}_3$ but are conjugated in $\text{Cr}_2$. On the other hand, different actions of $\text{PSL}_2(F_7)$ are not conjugated in $\text{Cr}_2$. This settles the primitive actions.
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This settles the **primitive** actions.
In this case, $G$ contains an abelian subgroup $H$ of rank 2.
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This gives examples of nonbirational actions of $G$, if $H$ contains $\mathbb{Z}/n \oplus \mathbb{Z}/n$ with $n = 5$, $n \geq 7$.

There are also examples when the Reichstein–Youssin invariant does not distinguish the actions, but the $BC_2$-class does.
We can assume that
\[ G = C_n \times G', \quad n \geq 2, \]
where \( G' \subset \text{GL}_2(k) \) is a lift of \( \bar{G}' \subset \text{PGL}_2(k) \).
We can assume that

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where \( G' \subset \text{GL}_2(k) \) is a lift of \( \tilde{G}' \subset \text{PGL}_2(k) \).

- \( G' = C_m \): Actions of cyclic groups on \( \mathbb{P}^1 \) lift to \( \text{GL}_2 \). Thus we have an action of \( G := C_n \times C_m \) on \( \mathbb{P}^2 \). These are birational if and only if the determinants differ by \( \pm 1 \).
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- \( \bar{G}' = \mathcal{D}_m, \mathfrak{A}_4, \mathfrak{S}_4, \) or \( \mathfrak{A}_5 \). Let \( n \) be such that \( \varphi(n) \geq 3 \). Then \( G \) admits nonbirational actions on \( \mathbb{P}^2 \).
Let $\epsilon$ be a primitive character of $C_n$. Let $V$ be the 2-dimensional representation of $G'$ lifting $\bar{G}' \subset \text{PGL}_2(k)$, and $V_\epsilon := V \otimes \epsilon$. 
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with a fixed point $p$. 
Let $\epsilon$ be a primitive character of $C_n$. Let $V$ be the 2-dimensional representation of $G'$ lifting $\bar{G}' \subset \text{PGL}_2(k)$, and $V_\epsilon := V \otimes \epsilon$.

This gives a generically free action of $G$ on

$$\mathbb{P}^2 = \mathbb{P}(I \oplus V_\epsilon),$$

with a fixed point $p$. To bring the $G$-action into standard form, we need to blow up $p$; we extract the classes

$$(C_n, \bar{G}' \subset k(\mathbb{P}^1), (\epsilon)) + (C_n, \bar{G}' \subset k(\mathbb{P}^1), (-\epsilon)),$$

which are incompressible.
Choosing primitive $\epsilon \neq \pm \epsilon'$, we find that

$$[\mathbb{P}(V_\epsilon) \Leftrightarrow G] \neq [\mathbb{P}(V_{\epsilon'}) \Leftrightarrow G] \in \text{Burn}_2(G).$$
Cheltsov–Shramov (2019), Cheltsov–Sarikyan (2022)

Classification of $G$-rigid and $G$-solid $\mathbb{P}^3$, 

Let $G'$ be $S_4$; $A_5$; $\text{PSL}_2(\mathbb{F}_7)$; or $2$. Let $C_p$ be the cyclic group of prime order $p > 7$, and put $G := C_n \cdot G'$. Then there exist embeddings $G, \phi : PGL_4$ that are not conjugated in $Cr_3$. 


Classification of $G$-rigid and $G$-solid $\mathbb{P}^3$, in particular, those that don’t fix a point or leave invariant a pair of skew lines.
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Classification of $G$-rigid and $G$-solid $\mathbb{P}^3$, in particular, those that don’t fix a point or leave invariant a pair of skew lines.

Kresch-T. 2021

Let $G'$ be

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Let $C_p$ be the cyclic group of prime order $p > 7$, and put

$$G := C_n \times G'.$$

Then there exist embeddings

$$G \hookrightarrow \text{PGL}_4$$

that are not conjugated in $\text{Cr}_3$. 
Proof: Let $V$ be a 3-dimensional representation of $G'$. Consider faithful $G$-actions on $\mathbb{P}(I \oplus V_\epsilon)$, where $V_\epsilon := V \otimes \epsilon$, $\epsilon : C_p \to k^\times$. These actions have a $G$-fixed point.
Proof: Let $V$ be a $3$-dimensional representation of $G'$. Consider faithful $G$-actions on

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After blowing up, we extract incompressible symbols

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After blowing up, we extract incompressible symbols

$$(C_p, \tilde{G}' \subset k(\mathbb{P}(V)), (\epsilon)) + (C_p, \tilde{G}' \subset k(\mathbb{P}(V)), (-\epsilon)).$$

Therefore,

$$[\mathbb{P}(I \oplus V_\epsilon) \subset G] \neq [\mathbb{P}(I \oplus V_{\epsilon'}) \subset G] \in \text{Burn}_3(G),$$

provided

$$\epsilon \neq \pm \epsilon'.$$
Produce linear actions on $\mathbb{P}^3$ of

$$G = \mathbb{D}_4 \times \mathbb{D}_5,$$

(dihedral groups of order 8 and 10), that leave a pair of two skew lines invariant and are not equivariantly birational.
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We use

$$\mathcal{BC}_3(G).$$
Summary

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  \[ \mathcal{B}_n(G), \quad \mathcal{B}C_n(G), \quad \text{Burn}_n(G) \]
  offer a new approach.