Divisorial stability: openness and cscK metrics

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with S. Boucksom
Overview and references

- Consider a projective complex manifold $X$ of dim $n$ with an ample line bundle $L$.
- Motivating question: $\exists$ cscK metric $\alpha \in c_1(L)$:
  \[ \text{Ric}(\alpha) \wedge \alpha^{n-1} = \text{const} \alpha^n. \]
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- Goal of talk: develop a notion of divisorial stability of a numerical class $\omega \in \text{Amp}(X)$ s.t.
  (a) $c_1(L)$ divisorially stable $\implies \exists!$ cscK metric in $c_1(L)$;
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- Here (a) relies on deep work by C. Li and Chen–Cheng. Conjecturally, converse is true.

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Tool: pluripotential theory on the Berkovich analytification of \( X \). References:

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Variational approach

- Existence of cscK metric determined by Mabuchi functional

\[ M : \mathcal{H} \to \mathbb{R}, \]

where \( \mathcal{H} = \{ \text{Kähler potentials wrt reference Kähler form} \}. \)
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  \[ \exists! \text{ cscK metric} \iff M \geq \varepsilon \| \cdot \| - C \quad \text{on } \mathcal{H} \]
  where \( \| \cdot \| \) is a “norm” on \( \mathcal{H} \), e.g. the functionals \( I, J, I - J, \ldots \).
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\[ \beta : \{ \text{volume forms on } X \text{ of mass 1} \} \to \mathbb{R}. \]

Same for \( \| \cdot \| \). The space of volume forms is independent of \( L \)!
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- Our stability notion mimics this when \( \mathbb{C} \) is *trivially valued* (hence non-Archimedean).
Related work

• Dervan '16 and Boucksom–Hisamoto–J '17 introduced uniform K-stability.
  By BHJ '19:
  \[ \exists! \text{cscK metric in } c_1(L) = \Rightarrow (X, L) \text{ uniformly K-stable.} \]

• Chi Li '20 introduced uniform K-stability for filtrations (see later) and proved:
  \[ (X, L) \text{ uniformly K-stable for filtrations } = \Rightarrow \exists! \text{cscK metric in } c_1(L). \]

• Goal of our work:
  give a new perspective on Li's stability notion;
  prove it is an open condition on the numerical class \( c_1(L) \);
  comment on the reverse implications.

• Other sources of inspiration:
  The Fano case (. . . , Berman–Boucksom–J '21, Li '22, . . . ).
  Valuative stability (Dervan–Legendre, Yaxiong Liu).
  Divisorial stability in the sense of Fujita.
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K-stability for filtrations I

• From now on:
  • \( X/\mathbb{C} = \) normal projective variety (or pair) with klt sings, of dimension \( n \);
  • \( L = \) ample \( \mathbb{Q} \)-line bundle on \( X \).

K-stability is typically defined using (ample) test configurations [Tian, Donaldson].

For example, \((X, L)\) is K-semistable iff \( M(X, L) \geq 0 \) for all tcs \((X, L)\).

Here \( M = \) Mabuchi functional (NA version \( \approx \) Donaldson–Futaki invariant).

• Can view a tc as a \( \mathbb{Z} \)-filtration \( F \) of finite type of \( \mathbb{R}(X, dL) \), \( d \) suff. div. . .

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• Idea [Székelyhidi]: use general \( \mathbb{Z} \)-filtrations \( \chi \). How to define \( M(\chi) \)?

• Let \( \chi_d = \) induced filtration on \( \mathbb{R}(X, dL) \) generated in degree 1.

• Székelyhidi used \( M(\chi) := \lim d M(\chi_d) \). Unclear if well-behaved. (More on this later.)

• If \( X \) smooth, Li (based on [BJ]) gave a definition of \( M(\chi) \) using NA pluripot theory.

• We give a alternative definition of \( M(\chi) \) that works also in the singular case.
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- Any tc $(\mathcal{X}, \mathcal{L})$ induces a finite atomic “Monge–Ampère” measure $\text{MA}(\mathcal{X}, \mathcal{L})$ on $X^{\text{div}}$. For example, if $\mathcal{X}$ normal with central fiber $\mathcal{X}_0 = \sum_i b_i E_i$, then

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- Our plan:
  - extend the MA operator from the space $\mathcal{T}$ of test configurations to the space $\mathcal{N}_\mathbb{R}$ of arbitrary $\mathbb{R}$-filtrations; this takes values in a class $\mathcal{M}^1$ of probability measures on $X^\text{an}$;
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  - define a functional $\beta: \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ such that
    \[
    \beta(\text{MA}(\mathcal{X}, \mathcal{L})) = M(\mathcal{X}, \mathcal{L}) \text{ for any tc } (\mathcal{X}, \mathcal{L}).
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The $\beta$-invariant on divisorial valuations

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- Well defined, by results of Lazarsfeld, L–Mustaţă, Boucksom–Favre–J.
- If $X$ is Fano and $L = -K_X$, then $\beta_L(F) = A(F)$ is (up to a constant) the functional considered by Fujita and Li. Moreover, in this case, $X$ $K$-semistable $\iff \beta(F) \geq 0$ for all $F$.
- Dervan–Legendre and Liu used $\beta(F)$ to define valuative stability.
- In general, don't expect to detect K-stability using divisorial valuations alone.
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- Set \( V = (L^n) \), and

\[
\|F\|_L = \frac{1}{V} \int_0^\infty \text{vol}(L - tF) \, dt > 0.
\]

Also denoted \( S(F) \) and called the expected vanishing order of \( \text{ord}_F \).

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- If \( X \) is Fano and \( L = -K_X \), then \( \beta_L(F) = A(F) - \|F\|_L \) is (up to a constant) the functional considered by Fujita and Li. Moreover, in this case, \( X \) K-semistable \( \iff \beta(F) \geq 0 \) for all \( F \).
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  Also denoted $S(F)$ and called the *expected vanishing order* of $\text{ord}_F$.
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  \[ \beta(F) := A(F) + \left. \frac{d}{dt} \right|_{t=0} \|F\|_{L+tK_X}. \]
The $\beta$-invariant on divisorial valuations

• First define $\beta$ on Dirac masses $\delta_v$, with $v = \text{ord}_F \in X^{\text{div}}$, $F$ prime divisor over $X$.
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- Dervan–Legendre and Liu used $\beta(F)$ to define valuative stability.
- In general, don’t expect to detect K-stability using divisorial valuations alone.
Divisorial measures and entropy

- A *divisorial measure* on $X$ is a convex combination

$$\mu = \sum_{i=1}^{N} m_i \delta_{v_i},$$

where $v_i = c_i \text{ord}_{F_i} \in X^{\text{div}}, m_i \in \mathbb{R}_{>0}, \sum_i m_i = 1$. 
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• The entropy functional $\text{Ent}: \mathcal{M} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is defined by

$$\text{Ent} (\mu) = \int_{X^{\text{an}}} A (v) \, d\mu (v),$$

where $A: X^{\text{an}} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is the largest lsc homogeneous function such that $A (\text{ord}_{F}) = A (F)$ for all prime divisors $F$ over $X$ [BFJ, JM, BdFFU, BJ].
The norm of a measure

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- Extend this to $E: C^0 \to \mathbb{R}$ by setting
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• If $(\mathcal{X}, \mathcal{L})$ is a tc, then $\|\text{MA}(\mathcal{X}, \mathcal{L})\|$ is the *minimum norm* in the sense of Dervan.
Measures of finite norm

- **Def**: the set of *measures of finite norm* is

  \[ \mathcal{M}^1 := \{ \mu \in \mathcal{M} \mid \| \mu \|_L < +\infty \} \subset \mathcal{M}. \]
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• **Thm.**
  · The space \( \mathcal{M}^1 \) and the strong topology do not depend on \( L \).
  · \( \mathcal{M}^{\text{div}} \subset \mathcal{M}^1 \) is strongly dense.
  · For any \( \mu \in \mathcal{M}^1 \), the norm \( \|\mu\|_L \) only depends on the numerical class \( \omega = c_1(L) \), and

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extends uniquely to a differentiable function on \( \text{Amp}(X) \).
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    extends uniquely to a differentiable function on \( \text{Amp}(X) \).

- **Proof** involves many applications of the Cauchy–Schwartz inequality, ultimately deriving from the Hodge Index Theorem.
The $\beta$-invariant of a measure of finite norm

- For any $\omega \in \text{Amp}(X)$, define a functional $\beta_\omega : M^1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\beta_\omega(\mu) = \text{Ent}(\mu) + \frac{d}{dt} \bigg|_{t=0} \|\mu\|_{\omega-tc_1(X)}.$$
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- **Thm.** The function $\omega \mapsto \beta_\omega(\mu)$ is continuous. Now fix an ample $\mathbb{Q}$-line bundle $L$.
  
  - For any prime divisor $F$ over $X$,
    $$\beta_L(\delta_{\text{ord}_F}) = \beta_L(F).$$
  
  - For any test configuration $(\mathcal{X}, \mathcal{L})$,
    $$\beta_L(\text{MA}(\mathcal{X}, \mathcal{L})) = M(\mathcal{X}, \mathcal{L}),$$
  
  the Mabuchi functional as defined by [Boucksom–Hisamoto–J].
Divisorial stability

• **Thm/Def.** The *divisorial stability threshold* is

\[ \sigma_{\text{div}}(\omega) := \sup \{ t \in \mathbb{R} \mid \beta_\omega(\mu) \geq t\|\mu\|_\omega \text{ for all } \mu \in \mathcal{M}^{\text{div}} \} \]

\[ = \sup \{ t \in \mathbb{R} \mid \beta_\omega(\mu) \geq t\|\mu\|_\omega \text{ for all } \mu \in \mathcal{M}^1 \}, \]

• where the second equality follows from strong density of \( \mathcal{M}^{\text{div}} \) in \( \mathcal{M}^1 \).

• **Cor.** \( \omega \mapsto \sigma_{\text{div}}(\omega) \) is continuous.

• **Def.** We say that:

  · \((X, \omega)\) is divisorially semistable if \( \sigma_{\text{div}}(\omega) \geq 0 \);
  
  · \((X, \omega)\) is (uniformly) divisorially stable if \( \sigma_{\text{div}}(\omega) > 0 \).

• **Cor.** Divisorial stability is an open condition on \( \text{Amp}(X) \).

• **Cor.** \((X, L)\) divisorially stable = \( (X, L)\) uniformly K-stable. (More later.)

• **Q.** How does this relate to uniform K-stability for filtrations?
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K-stability for filtrations

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K-stability for filtrations

- Fix an ample $\mathbb{Q}$-line bundle $L$.
- Given a filtration $\chi \in N_{\mathbb{R}}$, define $\chi_d \in T$ as the $\mathbb{Z}$-filtration on $R(X, dL)$ generated in degree 1 by $\lfloor \chi \rfloor$ on $H^0(X, dL)$.
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- **Thm.** The limit $MA(\chi) := \lim_d MA(\chi_d)$ exists in $\mathcal{M}^1$. The resulting map

$$MA : \mathcal{N}_\mathbb{R} \rightarrow \mathcal{M}^1$$

is not surjective in general, but its image contains $\mathcal{M}^{\text{div}}$.

- **Rmk.** The space of not necessarily ample test configurations for $(X, L)$ defines a subset of $\mathcal{N}_\mathbb{R}$ whose image under $MA$ equals $\mathcal{M}^{\text{div}}$.
- **Def.** Define $M(\chi) := \beta(MA(\chi))$ for $\chi \in \mathcal{N}_\mathbb{R}$. Say $(X, L)$ is uniformly K-stable for filtrations if $\exists \varepsilon > 0$ such that $M(\chi) \geq \varepsilon \|\chi\|$. 
- **Advantage.** The divisorial stability notion: the set $\mathcal{M}^{\text{div}}$ does not depend on $L$. 

**Cor.** [Li] Divisorial stability is equivalent to uniform K-stability for filtrations.
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- Define \( M(\chi) := \beta(MA(\chi)) \) for \( \chi \in \mathcal{N}_\mathbb{R} \). Say \( (X, L) \) is *uniformly K-stable for filtrations* if \( \exists \varepsilon > 0 \) such that \( M(\chi) \geq \varepsilon \| \chi \| \).
K-stability for filtrations

- Fix an ample \(\mathbb{Q}\)-line bundle \(L\).
- Given a filtration \(\chi \in \mathcal{N}_\mathbb{R}\), define \(\chi_d \in \mathcal{T}\) as the \(\mathbb{Z}\)-filtration on \(R(X, dL)\) generated in degree 1 by \([\chi]\) on \(H^0(X, dL)\).
- Thm. The limit \(\text{MA}(\chi) := \lim_d \text{MA}(\chi_d)\) exists in \(\mathcal{M}^1\). The resulting map

\[
\text{MA}: \mathcal{N}_\mathbb{R} \rightarrow \mathcal{M}^1
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- Cor [Li]. Divisorial stability is equivalent to uniform K-stability for filtrations.
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- Rmk. The space of not necessarily ample test configurations for $(X, L)$ defines a subset of $\mathcal{N}_\mathbb{R}$ whose image under $\text{MA}$ equals $\mathcal{M}^{\text{div}}$.
- Define $M(\chi) := \beta(\text{MA}(\chi))$ for $\chi \in \mathcal{N}_\mathbb{R}$. Say $(X, L)$ is uniformly K-stable for filtrations if $\exists \varepsilon > 0$ such that $M(\chi) \geq \varepsilon \|\chi\|$
- Cor [Li]. Divisorial stability is equivalent to uniform K-stability for filtrations.
- Advantage of the divisorial stability notion: the set $\mathcal{M}^{\text{div}}$ does not depend on $L$. 
Divisorial stability and uniform K-stability

• Again fix an ample $\mathbb{Q}$-line bundle $L$. 
Divisorial stability and uniform K-stability

- Again fix an ample \( \mathbb{Q} \)-line bundle \( L \).
- Any tc \( (X, L) \) for \( (X, L) \) induces a divisorial measure \( \mu = \text{MA}(X, L) \in \mathcal{M}^{\text{div}} \).
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• As we have seen:
  · \( \|\mu\|_L = \|(\mathcal{X}, L)\| \), the minimum norm in the sense of Dervan;
  · \( \beta_L(\mu) \) is the Mabuchi functional by B–Hisamoto-J;
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• We conjecture that the converse holds. This would follow from:

• $Q$. Given a non-ample tc $(X, L)$, let $\chi$ be the associated norms, and $(\chi_{d})$ be the net of canonical approximants in $T$. Do we have $\lim_{d} \text{Ent}(MA(\chi_{d})) = \text{Ent}(MA(\chi))$?
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  **Entropy regularization conjecture.** For every $\mu \in \mathcal{M}^{\text{div}}$ there exists a sequence $(X_m, L_m)$ of tcs such that $\mu_m := \text{MA}(X_m, L_m)$ converges strongly to $\mu$, and

  \[
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Divisorial stability and uniform K-stability

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Divisorial stability and cscK metrics

- Now assume $X$ is smooth and $L$ ample. To summarize, we have

$$(X, L) \text{ divisorially stable } \iff (X, L) \text{ uniformly K-stable for filtrations}$$

$$(X, L) \text{ uniformly K-stable } \iff M^{\text{Arch}} \text{ coercive}$$

$\exists! \text{ cscK metric in } c_1(L)$$

Does uniform K-stability imply divisorial stability (see above)?

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THANK YOU!