An embedding problem for closed 3-forms on 5-manifolds

Fabian Lehmann

SCGP

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Let $Z$ be a complex 3-fold with holomorphic volume form $\Psi + i \hat{\Psi}$. Hitchin (2000) gave a description of this geometric structure purely in terms of the real part $\Psi$.

- $\Psi$ is a stable 3-form (like the 3-form defining a $G_2$-structure): at each tangent space, $\Psi$ has an open orbit under the action of $GL(6, \mathbb{R})$ with stabiliser $SL(3, \mathbb{C})$. 

[Note: The text is a bit cut off, but the extracted content is readable and does not require any hallucination.]
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- $\Psi$ is a stable 3-form (like the 3-form defining a $G_2$-structure): at each tangent space, $\Psi$ has an open orbit under the action of $GL(6, \mathbb{R})$ with stabiliser $SL(3, \mathbb{C})$.
- The imaginary part $\hat{\Psi}$ is completely determined by $\Psi$.
- Given $d\Psi = 0$, the condition $d\hat{\Psi} = 0$ means that $\Psi$ is a critical point in its cohomology class for a certain functional on 3-forms.

On a 7-manifold with boundary a $G_2$-structure $\phi$ induces a stable 3-form on the boundary. What is the structure inherited by the boundary $\partial Z$ (or more generally a hypersurface $M \subset Z$) of a complex 3-fold with holomorphic volume form $\Psi + i\hat{\Psi}$?

The restriction of $\Psi$ induces a closed 3-form on a real hypersurface $M \subset Z$. 

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The restriction of $\Psi$ induces a closed 3-form on a real hypersurface $M \subset Z$. 
This leads us to consider the two questions:

1. What is the structure induced on a 5-manifold by a closed 3-form $\psi$?
2. Given a closed 3-form $\psi$ on $M$, can we find an embedding $F: M \rightarrow Z$ such that $F^* \Psi = \psi$ ("$F$ realises $\psi$")?
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This gave rise to the influential embedding problem in CR-geometry: Which abstract CR-structures can be realised by an embedding?
A closed 3-form in five dimensions locally depends on
\[ \dim \Lambda^2 - \dim \Lambda^1 + \dim \Lambda^0 = 10 - 5 + 1 = 6 \]
functions:
- \( \psi = d\tau \) for a 2-form \( \tau \) ⇒ add \( \dim \Lambda^2 \).
- \( \tau + d\eta \) gives same 3-form ⇒ subtract \( \dim \Lambda^1 \).
- If \( \eta = df \), \( \tau \) does not change ⇒ add \( \dim \Lambda^0 \).
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So up to diffeomorphism a closed 3-form depends on $6 - 5 = 1$ function.
Analogy with Riemannian metrics on surfaces

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So up to diffeomorphism a closed 3-form depends on \( 6 - 5 = 1 \) function.
A similar count for Riemannian metrics on 2-dimensional manifolds gives \( 3 - 2 = 1 \).
Weyl embedding problem

- Weyl (1915): Given a Riemannian metric on $S^2$ with positive Gaussian curvature, is there an isometric embedding into $\mathbb{R}^3$? Proved by Nirenberg and Pogorelov.
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- Positive Gaussian curvature means that the image $F(S^2)$ bounds a strongly convex subset of $\mathbb{R}^3$. 

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- A useful condition in the theory of several complex variables is the weaker notion of (strong) pseudo-convexity.
Definition

$\psi \in \Omega^3(M)$ is called strongly pseudoconvex if it satisfies the following three properties:

1. The skew-symmetric bilinear form

\[
\Lambda^1 \times \Lambda^1 \rightarrow \Lambda^5, \quad (\lambda, \eta) \mapsto \lambda \wedge \eta \wedge \psi
\]  

has maximal rank, i.e. 4, at each point. This defines a rank 4 subbundle $H \subset TM$.

2. Let $\theta$ be a 1-form which at each point spans the 1-dimensional kernel of (0.1). Then

$\psi = \theta \wedge \alpha$ for some 2-form $\alpha$ and $H$ is the kernel of $\theta$. 

In particular, $\theta \wedge (d\theta)^2$ is a positive multiple of $\theta \wedge \alpha^2$. Therefore, $d\theta$ is non-degenerate on $H$, which means that $H$ is a contact structure.
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Normalisation

The decomposition $\psi = \theta \wedge \alpha$ is not unique. We can

- replace $\theta$ by $f\theta$ and $\alpha$ by $f^{-1}\alpha$, where $f$ is a non-vanishing function.

Normalise $\theta$ by scaling by some function $f$ such that $\theta \wedge \alpha^2 = \theta \wedge (d\theta)^2$. This fixes preferred contact 1-form $\theta$ and Reeb vector field $v$, which is defined by $\theta(v) = 1$ and $v \lrcorner d\theta = 0$.

This gives decomposition $TM = \mathbb{R}v \oplus H$ and $\Omega^p(M) = \theta \wedge \Omega^{p-1}H \oplus \Omega^pH$, where $\gamma \in \Omega^pH$ iff $v \lrcorner \gamma = 0$.

We have $\omega := d\theta \in \Omega^2H$. Normalise $\alpha$ by requiring $\alpha \in \Omega^2H$. 

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- add $\theta \wedge \chi, \chi \in \Omega^1(M)$, to $\alpha$. 

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where $\gamma \in \Omega^p_H$ iff $\nu \perp \gamma = 0$.

We have $\omega := d\theta \in \Omega^2_H$. Normalise $\alpha$ by requiring $\alpha \in \Omega^2_H$. 
The Lie derivative $L_v$ preserves the decomposition and the exterior derivative $d$ splits on $\Omega^p_H$ as

$$d_H : \Omega^p_H \to \Omega^{p+1}_H$$

and

$$\theta \wedge L_v : \Omega^p_H \to \theta \wedge \Omega^p_H.$$
$d\psi = 0$ is equivalent to

$$\omega \wedge \alpha = \theta \wedge d\alpha,$$

which is equivalent to

$$\omega \wedge \alpha = 0, \quad d_H \alpha = 0.$$

Normalisation gives $\omega^2 = \alpha^2$. Thus $(H, \omega + i\alpha)$ is an integrable CR-structure. $(M, v, H, \omega, \alpha)$ is an $\text{SL}(2, \mathbb{C})$-structure. Contact analogue of complex 2-fold with trivial canonical bundle.
Suppose $\psi$ induced by embedding $F : M \to (Z, \Psi + i\Psi)$. Then $\hat{\psi} = F^* \hat{\Psi}$ decomposes as $\theta \wedge \beta$, where $\beta \in \Omega^2_H$ satisfies

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Thus $(M, \nu, H, \omega, \alpha, \beta)$ is a nearly hypo $SU(2)$-structure (contact version of hyperkähler 4-manifold).
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Conti–Salamon (2005) have studied those in the context of real hypersurfaces in torsion-free SU(3)-manifolds.
We can think of the realisation problem for a strongly pseudoconvex 3-form $\psi$ as consisting of two parts:

Problem 1: Find $\beta \in \Omega^2_{\text{orth}}$ orthogonal to $\omega$ and $\alpha$, and of unit length, which solves $dH\beta = 0$.

This problem can be seen as a contact version of the Calabi problem in dimension four.

Problem 2: Find an embedding for the strongly pseudoconvex CR-manifold $(M, H, \alpha + i\beta)$. 
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Given $\psi$, an embedding $F : M \hookrightarrow Z$ with $F^*\Psi = \psi$ is in general not unique.
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If $F(M)$ is the boundary of a domain $U \subset Z$, and if $\Phi : \bar{U} \rightarrow Z$ is a diffeomorphism to its image which is holomorphic and satisfies $\Phi^*\Psi = \Psi$, then $\Phi \circ F$ is another embedding which realises $\psi$. 

If for example the ambient space is $C^3$, then the restriction of every element in $SL(3, C)$ is such a diffeomorphism.
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Is embedding unique up to such diffeomorphisms?
For an $SU(2)$-structure as above $(\omega, \alpha, \beta)$ is an orthonormal triple for the wedge product pairing

$$(\sigma \cdot \tau) \text{Vol}_H = \sigma \wedge \tau,$$

with volume form $\text{Vol}_H = \frac{1}{2} \omega^2$ on $H$. 

(anti)-self-dual 2-forms
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$$\Lambda^2_H = \Lambda^+_H \oplus \Lambda^-_H$$

into self-dual and anti-self-dual 2-forms on $H$. 
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Set

$$\mathcal{H} = \{ \sigma \in \Omega^-_H : d_H \sigma = 0 \}.$$
Theorem (Donaldson-L., 2022)

Let \((v, H, \omega, \alpha, \beta)\) be a nearly hypo SU(2)-structure on a closed 5-manifold with \(H^3(M, \mathbb{R}) = 0\).
Theorem (Donaldson-L., 2022)

Let \((v, H, \omega, \alpha, \beta)\) be a nearly hypo SU(2)-structure on a closed 5-manifold with \(H^3(M, \mathbb{R}) = 0\).

- The space \(\mathcal{H}\) is finite dimensional.
Results

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- The space \(\mathcal{H}\) is finite dimensional.
- Suppose the structure is induced by an embedding \(F : M \hookrightarrow Z\) and \(\mathcal{H} = 0\). Then all closed 3-forms \(\tilde{\psi}\) sufficiently close to \(\psi = F^*\Psi\) can be realised by an embedding \(\tilde{F}\) close to \(F\). If the ambient space is \(Z = \mathbb{C}^3\), then \(\tilde{F}\) in a neighbourhood of \(F\) is unique up to holomorphic diffeomorphisms as above.
Theorem (Donaldson-L., 2022)

Let $(v, H, \omega, \alpha, \beta)$ be a nearly hypo SU(2)-structure on a closed 5-manifold with $H^3(M, \mathbb{R}) = 0$.

- The space $\mathcal{H}$ is finite dimensional.
- Suppose the structure is induced by an embedding $F : M \hookrightarrow Z$ and $\mathcal{H} = 0$. Then all closed 3-forms $\tilde{\psi}$ sufficiently close to $\psi = F^*\Psi$ can be realised by an embedding $\tilde{F}$ close to $F$. If the ambient space is $Z = \mathbb{C}^3$, then $\tilde{F}$ in a neighbourhood of $F$ is unique up to holomorphic diffeomorphisms as above.
- If the SU(2)-structure is Sasaki–Einstein and $Z$ is Stein, then $\mathcal{H} = 0$. In particular this is true for the standard embedding $S^5 \hookrightarrow \mathbb{C}^3$. 
Linearised problem

Denote by $\mathcal{E}(M, Z)$ set of smooth embeddings of $M$ into $Z$. 
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Linearisation at $F$:

$$DP(F) : C^\infty(M, F^*TZ) \to d\Omega^2(M),$$

$$V \mapsto L_V\psi = d(V \psi).$$
We have

\[ F^*TZ = \mathbb{R}v \oplus \mathbb{R}Iv \oplus H, \]

where \( I \) is complex structure of ambient space.
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Define

$$K(F) : C^\infty(M) \oplus C^\infty(M) \oplus C^\infty(M, H) \to \Omega^2(M)
\quad (f, g, w) \mapsto (fv + glv + w) \psi = f\alpha + g\beta + (w\alpha) \wedge \theta.$$
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\[ F^* T_Z = \mathbb{R} v \oplus \mathbb{R} I v \oplus H, \]

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\[ K(F) : \mathcal{C}^\infty(M) \oplus \mathcal{C}^\infty(M) \oplus \mathcal{C}^\infty(M, H) \rightarrow \Omega^2(M) \]

\[ (f, g, w) \mapsto (fv + glv + w) \downarrow \psi = f \alpha + g \beta + (w \downarrow \alpha) \wedge \theta. \]

Then \( DP(F) = d \circ K(F) \). Thus \( DP(F) \) is surjective onto \( d\Omega^2(M) \) iff

\[ \text{im} K(F) = \Gamma(\text{span}\{\alpha, \beta\} \oplus \Lambda^1_H \wedge \theta) = \Omega^2(M)/d\Omega^1(M). \]
Solving linearised problem

- Given $\sigma \in \Omega^2(M)$, we need to add an exact 2-form to eliminate the component of $\sigma$ in $\mathbb{R}\omega \oplus \Lambda_H^-$. 
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- Given $\sigma \in \Omega^2(M)$, we need to add an exact 2-form to eliminate the component of $\sigma$ in $\mathbb{R}\omega \oplus \Lambda^-_H$.
- Decompose

$$\sigma = \sigma^+ + \sigma^- + \chi \wedge \theta,$$

where $\sigma^{\pm} \in \Omega^\pm_H$ and $\chi \in \Omega^1_H$. 

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- Decompose

$$\sigma = \sigma^+ + \sigma^- + \chi \wedge \theta,$$

where $\sigma^\pm \in \Omega^\pm_H$ and $\chi \in \Omega^1_H$.  
- To eliminate $\sigma^-$, we need to solve the equation

$$d^-_H \eta = \sigma^-,$$

where $d^-_H = \pi^- \circ d_H : \Omega^1_H \to \Omega^-_H$.  

Solving the linearised problem

- Suppose $\eta \in \Omega^1_H$ solves

\[
d \eta = \sigma.
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- Suppose $\eta \in \Omega^1_H$ solves
  \[
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  \]

- Then
  \[
  \sigma - d \eta = \tilde{\sigma}^+ + \chi' \wedge \theta = h_\alpha \alpha + h_\beta \beta + h_\omega \omega + \chi' \wedge \theta.
  \]
Solving the linearised problem

Suppose $\eta \in \Omega^1_H$ solves

$$d_H^- \eta = \sigma^-.$$

Then

$$\sigma - d\eta = \tilde{\sigma}^+ + \chi' \wedge \theta = h_\alpha \alpha + h_\beta \beta + h_\omega \omega + \chi' \wedge \theta.$$

Then

$$\sigma - d\eta - d(h_\omega \theta) = h_\alpha \alpha + h_\beta \beta + \chi'' \wedge \theta \in \text{im}K(F).$$
The $d^{-}_H$-equation

- We can solve linearised problem iff

$$d^{-}_H : \Omega^1_H \to \Omega^{-}_H$$

is surjective.
We can solve linearised problem iff

\[ d_H^- : \Omega^1_H \to \Omega^-_H \]

is surjective.

We have

\[ (d_H^-)^* = d^*_H = -\ast_H d_H \ast_H. \]
The $d_H^-$-equation

- We can solve linearised problem iff

$$d_H^- : \Omega_H^1 \to \Omega_H^-$$

is surjective.

- We have

$$(d_H^-)^* = d_H^* = - *_H d_H *_H.$$ 

- Thus \( \text{coker } d_H^- = \mathcal{H} = \{ \sigma \in \Omega_H^- : d_H \sigma = 0 \}.$$
In 4-dimensional Riemannian geometry:

\[ \{ \sigma \in \Omega^- : d\sigma = 0 \} = \{ \sigma \in \Omega^- : d\sigma = 0, d^* \sigma = 0 \} = \ker \Delta_{|\Omega^-}. \]

Laplace operator \( \Delta_{|\Omega^-} \) elliptic, so kernel finite dimensional.
In 4-dimensional Riemannian geometry:

$$\{ \sigma \in \Omega^- : d\sigma = 0 \} = \{ \sigma \in \Omega^- : d\sigma = 0, d^*\sigma = 0 \} = \ker \Delta|_{\Omega^-}.$$ 

Laplace operator $\Delta|_{\Omega^-}$ elliptic, so kernel finite dimensional.

Set $\Delta_H = d_H d_H^* + d_H^* d_H$ and $\Box_H = d_H^* (d_H^*)^*$. Then $\Box_H = \frac{1}{2} \Delta_H|_{\Omega_H^-}$. 

□$H$ is not elliptic! Missing derivatives in $v$ direction. In particular, no coercive estimate of the form $\|\sigma\|_2 \lesssim (\Box_H \sigma, \sigma) + \|\sigma\|_2$. 
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- Set \( \Delta_H = d_H d_H^* + d_H^* d_H \) and \( \Box_H = d_H^-(d_H^-)^* \). Then \( \Box_H = \frac{1}{2} \Delta_H|_{\Omega^-_H}. \)

- \( \Box_H \) is not elliptic! Missing derivatives in \( \nu \) direction. In particular, no coercive estimate of the form

\[ \|\sigma\|_{1}^2 \lesssim (\Box_H \sigma, \sigma) + \|\sigma\|^2. \]
Sub-ellipticity

- $g = \theta^2 + g_H$ is Riemannian metric on $M$. Levi-Civita connection does not preserve $H$.

At the cost of introducing torsion one can find partial covariant derivative $\nabla$ which looks like a Levi-Civita connection for $g_H$:

$\nabla$ preserves $H$, $\nabla g_H = 0$ and torsion has no component in $\Lambda^2 H \otimes H$.

Then

$\Delta H = \nabla^* H \nabla H + I \circ \nabla v + R \nabla$

where $I$ is CR-structure given by $\alpha + i \beta$.

$v$ is locally commutator of sections of $H$ (so two $H$-derivatives!). This gives "1/2"-estimate

$\|\sigma\|_2^{1/2} \lesssim (\nabla^* H \nabla H \sigma, \sigma) + \|\sigma\|_2$. 
Sub-ellipticity

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$$\Delta_H = \nabla^*_H \nabla_H + I \circ \nabla_v + R^{\nabla}$$

$$= - (\nabla^2 e_1 + \cdots + \nabla^2 e_4) + I \circ \nabla_v + R^{\nabla},$$

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Sub-ellipticity

- $g = \theta^2 + g_H$ is Riemannian metric on $M$. Levi-Civita connection does not preserve $H$.
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\Delta_H = \nabla^* H \nabla_H + I \circ \nabla_\nu + R^\nabla
= -(\nabla^2_{e_1} + \cdots + \nabla^2_{e_4}) + I \circ \nabla_\nu + R^\nabla,
\]

where $I$ is CR-structure given by $\alpha + i\beta$.
- $\nu$ is locally commutator of sections of $H$ (so two $H$-derivatives!). This gives “$1/2$”-estimate

\[
\|\sigma\|^{2^{1/2}} \lesssim (\nabla^* H \nabla_H \sigma, \sigma) + \|\sigma\|^2.
\]
By choice of the connection, $\nabla_v$ preserves $\Omega_H^-$. Thinking of $\Lambda^2_H$ as skew-symmetric endomorphisms of $H$, the action of $I$ is the commutator $[I, \cdot]$. Every orientation preserving element in $\mathfrak{so}(H)$ commutes with every orientation reversing element if $\mathfrak{so}(H)$. Thus $I \circ \nabla_v = 0$ on $\Omega_H^-$. 

$\Box$ is sub-elliptic and by the theory of sub-elliptic operators $H = \ker \Box_H$ is finite-dimensional. 

On $\Omega_H^+$, $\Delta_H$ is not sub-elliptic.
By choice of the connection, $\nabla_v$ preserves $\Omega_H^-$. Thinking of $\Lambda^2_H$ is as skew-symmetric endomorphisms of $H$, the action of $I$ is the commutator $[I, \cdot]$. Every orientation preserving element in $\mathfrak{so}(H)$ commutes with every orientation reversing element if $\mathfrak{so}(H)$. Thus $I \circ \nabla_v = 0$ on $\Omega_H^-$. Thus

$$\|\sigma\|_2 \lesssim (\square_H \sigma, \sigma) + \|\sigma\|^2.$$

$\square_H$ is sub-elliptic and by the theory of sub-elliptic operators $\mathcal{H} = \ker \square_H$ is finite-dimensional.
By choice of the connection, $\nabla_v$ preserves $\Omega^-_H$. Thinking of $\Lambda^2_H$ is as skew-symmetric endomorphisms of $H$, the action of $I$ is the commutator $[I, \cdot]$. Every orientation preserving element in $\mathfrak{so}(H)$ commutes with every orientation reversing element if $\mathfrak{so}(H)$. Thus $I \circ \nabla_v = 0$ on $\Omega^-_H$.

Thus

$$\|\sigma\|_2^2 \lesssim (\Box_H \sigma, \sigma) + \|\sigma\|^2.$$  

$\Box_H$ is sub-elliptic and by the theory of sub-elliptic operators $\mathcal{H} = \ker \Box_H$ is finite-dimensional.

On $\Omega^+_H$, $\Delta_H$ is not sub-elliptic.
Nash–Moser–Hamilton theory

Hamilton (1977) worked on deformation theory of complex manifolds with boundary. His uniform higher order estimates can be applied to our uniform “1/2”-estimate to prove that $H = 0$ is an open condition, apply the Nash–Moser inverse function theorem.
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- prove that $\mathcal{H} = 0$ is an open condition,
Hamilton (1977) worked on deformation theory of complex manifolds with boundary. His uniform higher order estimates can be applied to our uniform “1/2”-estimate to

- prove that $\mathcal{H} = 0$ is an open condition,
- apply the Nash–Moser inverse function theorem.
On a CR-manifold one has a \( \bar{\partial}_b \)-operator which is the abstraction of the restriction of \( \bar{\partial} \).

\[ \Lambda_H^- \subset \Lambda^{1,1}_1 M. \]

If \( M \subset Z \) is strongly pseudoconvex and \( Z \) is Stein, then

\[ \{ \sigma \in \Omega^{1,1}_1 : \bar{\partial}_b \sigma = 0, \bar{\partial}^*_b \sigma = 0 \} = 0. \]

\( d_H \sigma = 0 \) for \( \sigma \in \Omega^-_H \) always implies \( \bar{\partial}^*_b \sigma = 0. \)

If the nearly hypo structure is Sasaki–Einstein (essentially means \( \beta = L_\nu \alpha \)), then \( \bar{\partial}_b \sigma = 0 \) if \( \sigma \in \mathcal{H} \).
Thank you!