THE ROLE OF INTERNAL WAVE TURBULENCE IN THE OCEANIC ENERGY PATHWAYS

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1 Introduction
   - Variability of the oceanic internal wavefield
   - Impact on climate

2 Wave-Wave Interactions: Theory / Observational Evidence

3 Results

4 Conclusions
1 Introduction
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Frequency spectra of oceanic kinetic energy

(Ferrari and Wunsch '09)

Inertial frequency: $$f = 1 \text{cpd} \times 2 \sin(\text{latitude})$$

Buoyancy frequency: $$N = \sqrt{-\frac{g}{\rho} \frac{\partial \rho}{\partial z}} (O(\text{cph}))$$

- $$\omega < f$$: Geostrophic/2D turbulence
- $$f < \omega < N$$: Internal wave turbulence
- $$\omega > N$$: instabilities, 3D turbulence
Some definitions to start with (1): Types of spectra

Notation: $\omega$ is the frequency, $\mathbf{p}$ is the 3D wavenumber; $k$ and $m$ are the magnitudes of horizontal and vertical wavenumbers

**Dispersion relation:**

\[
\left( \frac{k}{m} \right)^2 = \frac{\omega^2 - f^2}{N^2 - \omega^2}
\]

Under **spatial homogeneity** and **horizontal isotropy**, with $a_p$ the normal variables of the linearized internal-wave equation

- **3D action density** (Wave Kinetic Equation): $n_p$:

  \[
  \langle a_p a_p^* \rangle = n_p \delta(\mathbf{p} - \mathbf{p}')
  \]

- **2D energy density** (e.g. Garrett and Munk):

  \[
  e(\omega, m) = \omega n(\omega, m) = 4\pi k \omega \left( \frac{\partial \omega}{\partial k} \right)^{-1} n_p
  \]
Some definitions to start with (2): 2D Fourier space

- 2D Fourier domain, in $\omega - m$ space or $k - m$ space (change coordinates using dispersion relation), physical boundaries:

  - 2 common approximations:
    - **Non-rotating**: $\omega \gg f$
    - **Hydrostatic**: $\omega \ll N$

Scale-invariant dispersion relation:

$$\omega(k, m) = N \frac{k}{m}$$
Garrett and Munk (1976): a universal spectrum?
spoiler alert: NO

4 versions. Cairns & Williams 1976 (GM76):

\[ e(\omega, m) = E_0 A(m) B(\omega), \]

\[ A(m) = \frac{2m_{*}^2}{\pi} \frac{1}{m_{*}^2 + m^2}, \]

\[ B(\omega) = \frac{2f}{\pi} \frac{1}{\omega \sqrt{\omega^2 - f^2}}. \]

figures from Polzin & Lvov, Rev. of Geophys. (2011)
Regional variability, vertical wavenumber

GM76 high frequency, high wavenumber “slopes”: \( e(\omega, m) \propto \omega^{-2} m^{-2} \)

Pollmann, JPO (2020): statistics of spectral slope in vertical wavenumber (over a decade of Argo data – West Pacific average between 18° and 22° N)
GM76 high frequency, high wavenumber “slopes”: $e(\omega, m) \propto \omega^{-2} m^{-2}$

*Le Boyer & Alford, JPO (2021):* statistics of **spectral slope in frequency**
(global database of $\sim 2000$ mooring time series – colors denote different ocean basins)
Time variability

lat = 31.5, lon = 25.1 W, depth = 3900 m

Mar 12, 1985

\(e(\omega)\) (W kg\(^{-1}\) cph)

\(\omega\) (cph)

(Off of analysis of global moorings by Le Boyer and Alford)
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Dense water sinking to the abyss at high latitudes must eventually rise.

Mechanism for upwelling: Turbulent diapycnal mixing of “dense bottom water”/“light water above”

Diapycnal diffusivity $K_D$ in advection-diffusion closure, Munk, Deep Sea Res. ’66:

\[ L = K_D / w \approx 1 \text{ km}, \quad \tau = K_D / w^2 \approx 200 \text{ y} \]
\[ \Rightarrow w = 1.2 \text{ cm/day}, \quad K_D = 1.3 \times 10^{-4} \text{ m}^2 / \text{s}, \]
with no turbulence it would be: $\tau \approx 1 \text{ M y}!!$

One of the main drivers of the Meridional Overturning Circulation (MOC)
\[ \Rightarrow \text{Climate is sensitive to global variability of } K_D \text{ (from global models)} \]

Melet et al, J. of Climate (2016)
Whalen et al, Nat. Rev. Earth & Env. (2020)

from Sévellec, Fedorov, Deep Sea Res. ’11
In stationary conditions, turbulent kinetic energy production rate:

\[ \mathcal{P} = \mathcal{B} + \epsilon \]

- **Mixing**: \( \mathcal{B} = R_f \mathcal{P} \), work of turbulent buoyancy fluxes against gravity. Diapycnal diffusivity: \( K_D \approx \frac{\mathcal{B}}{\rho N^2} \)

- **Dissipation**: \( \epsilon = (1 - R_f) \mathcal{P} \)

\( R_f \approx 0.17 \), **flux Richardson number** – Osborn 1980 parameterization

Finescale parameterization: semi-heuristic statement that \( \mathcal{P} \) is a function of the internal wave field:

\[ \mathcal{P} = \mathcal{P}_0 \frac{f}{f_0} \frac{N^2}{N_0^2} \frac{E^2}{E_0^2} \]

( Gregg ’89, Henyey ’91, Polzin et al. ’95)

- Scaling from dimensional analysis of **wave-wave interactions** for a GM76 wave field (\( f_0, N_0, E_0 \) are the reference GM76 values)

- Empirical constant \( \mathcal{P}_0 \approx 8 \times 10^{-10} \) W/kg, curve fitting of observations
Finescale parameterization: underlying interpretation

Same domain as in slide 3, but transposed

Schematic of finescale parameterization from Polzin et al. 14

- **Energy sources** at low-frequency, low wavenumbers: inertial oscillations (induced by wind forcing), tides, mesoscale eddies
- **Energy sink** at high wn: turbulence (past 10m: shear instability)
- Trick: Internal-wave energy flux toward small vertical scales = turbulent kinetic energy production $\mathcal{P}$
Global maps of dissipation and mixing

\[ \mathcal{P} = \mathcal{P}_0 \frac{f}{f_0} \frac{N^2}{N_0^2} \frac{E^2}{E_0^2} \]

Globally integrated \( \mathcal{P} \simeq 2 \text{ TW} \)

(1.5 TW int. tide, 0.4 TW near-inertial, 0.2 TW geostr. - lee waves)

Waterhouse et al., JPO (2014)

Finescale estimates of global patterns of mixing:

\[ K_D \simeq \frac{R_f}{\rho_0 N^2} \mathcal{P} \]

Whalen et al., JPO (2015)

Finescale estimates of global patterns of turbulent dissipation:

\[ \epsilon \simeq (1 - R_f) \mathcal{P} \]
Finescale vs microstructure

Microstructure profilers = “real measurements” of turbulence (from stress tensor correlations)

Direct comparison of finescale estimates (from ARGO profiles) and microstructure measurements, for different experiments (black rectangles in mixing map in previous slide)

Whalen et al., JPO (2015)
Mixing Parameterizations in GCMs

- Energy sources for internal waves, *generation “hot spots”*:
  - Internal tide (*Vic et al. 2019*)
  - Lee waves (*Nikurashin & Ferrari 2011*)

  1/3 of energy dissipated locally (near field), 2/3 of energy radiates away (far field)
  - **Near field**: *St Laurent and Garrett (2002), Polzin (2009)* semi-empirical (finescale-based) parameterization
  - **Far field**: low vertical modes cover over a thousand km before being completely dissipated. *De Lavergne et al. (2019)*, **drag term**: 60% is wave-wave interactions (*PSI*) + other effects, e.g. scattering with abyssal hills (*Bühler & Holmes-Cerfon (2011)* scaling)

A lot of empirical laws! Clearly, *internal wave resonances play major role*

figure from *Zaron 2019*
Motivation

Internal waves in the ocean have very complex phenomenology

Distribution of $K_D$ in global models affects the MOC
Central role of wave-wave resonant interactions to determine $K_D$, $\epsilon$
Transfers to dissipation poorly understood, Whalen et al, Nat. Rev. ’20

Candidates: (i) wave-wave interactions; (ii) topographic scattering and reflection (e.g. Dauxois & Young, JFM (1999)) ; (iii) interaction with background mean flow (e.g. Bühler & McIntyre, JFM (2005))

“Studies focused on these individual energy transfer processes ... would be valuable. Understanding of wave-wave interactions using a spectrum close to observations is especially needed.”
Highly recommended

First-hand account of the life of Walter Munk (1917-2019)
Springer (2010)

State of the art of research in ocean mixing
Elsevier (2021)
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Natural framework to cast wave-wave interactions

- Originates in the ’60s from work by Klaus Hasselmann and Vladimir Zakharov (surface waves)
- The main object of the theory is the **Wave Kinetic Equation**
- For **internal waves**, first derived by Dirk Olbers in ’76, see **Müller et al ’86**

**Wave Kinetic Equation**

- Governing dynamics for the statistics in internal-wave band (figure), like a **turbulent inertial range**
- Here, ideally the other processes are negligible
- Irreversible transfers given by wave-wave resonant interactions
Simple Boussinesq approximation

5 equations in 5 variables

Velocity potential: $\phi$
Isopycnal layer thickness: $\Pi$

$$H = \frac{1}{2} \int \left( \Pi |\nabla \phi|^2 - g \left| \int^\rho \frac{\Pi - \Pi_0}{\rho_1} d\rho_1 \right|^2 \right) dx d\rho$$

2 (beautifully symmetric) equations in 2 scalar fields

Wave Kinetic Equation:
“standard step” in Wave Turbulence, but delicate mathematical proof

Assumptions:
- Isopycnal coordinates
- Horizontal isotropy
- Spatial homogeneity, constant $N$
- Zero total potential vorticity
- Hydrostatic balance

normal variables

$$a_p := \sqrt{\frac{\omega_p}{2g}} \frac{N}{k} \Pi_p - i \sqrt{\frac{g}{2\omega N}} \Phi_p$$

Assumptions:
- Random Gaussian field
- Weak nonlinearity

$$\langle a_p a^*_p \rangle = n_p \delta(p - p')$$
Wave Kinetic Equation for internal waves


\[
\frac{\partial n_p}{\partial t} = 4\pi \int \left( f_{pp_1p_2}^p |V_{pp_1p_2}^p|^2 \delta_{p+p_1-p_2} \delta_{\omega_p + \omega_{p_1} - \omega_{p_2}} - 2 f_{pp_1p_2}^p |V_{pp_1p_2}^p|^2 \delta_{p-p_1-p_2} \delta_{\omega_p - \omega_{p_1} - \omega_{p_2}} \right) dp_1 dp_2,
\]

R.h.s. is the **collision integral**: contains all wave-wave interactions

- \( f_{pp_1p_2}^p = n_{p_1} n_{p_2} - n_p (n_{p_1} + n_{p_2}) \), **quadratic in action density**
- \( V_{pp_1p_2}^p \) is the **matrix element**
- Dirac \( \delta \)'s impose triad resonance conditions (energy/momentum conservation)
Scale separated triads

These 3 classes of **nonlocal triads** still shape most of today’s understanding of internal-wave interactions!
PSI and Induced Diffusion approximations

- **p** large-scale, **p**₁ and **p**₂ small-scale, with \( \omega_p \sim 2\omega_p₁, 2\omega_p₂ \) and \( n_p \gg n_{p₁}, n_{p₂} \):

  \[
  \text{PSI:} \quad \frac{\partial n_p}{\partial t} \simeq -\left( 4\pi \int |V_{p₁, -p₁}^p|^2 \delta_{\omega_p - 2\omega_p₁} (n_{p₁} + n_{-p₁}) d\mathbf{p}_₁ \right) n_p
  \]

- **Exponential decay** of frequency \( \omega \) into 2 waves of frequency \( \omega/2 \).
- e.g. \( M₂ \) tide (T=12h25'), if \( \omega > 2f \iff \omega > \sin(\text{lat})/(12h) \iff |\text{lat}| < 29° \) predicts secondary peak of period 24h50' below 29°, correlated with \( M₂ \)

  - Taylor-expanding twice for small \( \mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1 \), Fokker-Planck:

    \[
    \text{ID:} \quad \frac{\partial n_p}{\partial t} \simeq \frac{\partial}{\partial p_j} \left( a_{ji}(\mathbf{p}) \frac{\partial}{\partial p_i} n_p \right), \quad \text{with}
    \]
    \[
    a_{ji}(\mathbf{p}) = 4\pi \int p_j p_i |V_{p₁, p₂}^p|^2 \delta_{\mathbf{p} - \mathbf{p}_₁ - \mathbf{p}_₂} \delta_{\omega_p - \omega_p₁ - \omega_p₂} n_{p₂} d\mathbf{p}_₁ d\mathbf{p}_₂
    \]

  - **Diffusion of action at high wn** mediated by low wn (in coefficient \( a_{ji} \))
PSI and Induced Diffusion approximations

- \( \mathbf{p} \) large-scale, \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) small-scale, with \( \omega_\mathbf{p} \approx 2\omega_{\mathbf{p}_1}, 2\omega_{\mathbf{p}_2} \) and \( n_\mathbf{p} \gg n_{\mathbf{p}_1}, n_{\mathbf{p}_2} \):

\[
\text{PSI:} \quad \frac{\partial n_\mathbf{p}}{\partial t} \simeq -\left( 4\pi \int |V^\mathbf{p}_{\mathbf{p}_1, -\mathbf{p}_1}|^2 \delta_{\omega_\mathbf{p} - 2\omega_{\mathbf{p}_1}} (n_{\mathbf{p}_1} + n_{-\mathbf{p}_1}) d\mathbf{p}_1 \right) n_\mathbf{p}
\]

- **Exponential decay** of frequency \( \omega \) into 2 waves of frequency \( \omega/2 \).
- e.g. \( M_2 \) tide (\( T=12h25' \)), if \( \omega > 2f \iff \omega > \sin(\text{lat})/(12h) \iff |\text{lat}| < 29^\circ \) predicts secondary peak of period 24h50' below 29°, correlated with \( M_2 \)

- Taylor-expanding twice for small \( \mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1 \), Fokker-Planck:

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\]

\[
a_{ji}(\mathbf{p}) = 4\pi \int p_j p_i |V^\mathbf{p}_{\mathbf{p}_1, \mathbf{p}_2}|^2 \delta_{\mathbf{p}-\mathbf{p}_1 - \mathbf{p}_2} \delta_{\omega_\mathbf{p} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}} n_{\mathbf{p}_2} d\mathbf{p}_1 d\mathbf{p}_2
\]

- **Diffusion of action at high \( \omega_\mathbf{p} \)** mediated by low \( \omega_\mathbf{p} \) (in coefficient \( a_{ji} \))
Observational evidence of wave-wave interactions

In oceanic observations:

*MacKinnon et al., JPO (2013)*
Bicoherence of the triad \((f, f, 2f)\) at 29°

Experiment conducted North of Hawaii where \(M_2\) tide is very strong

In high resolution numerical simulations:

*Pan et al., JPO (2020)*

Evaluation of collision integrand
Symmetric ID peaks at \(k_2 \to 0\) and \(k_2 \to k\)

In wave tank experiments:

*Grayson et al. (2022) (Cambridge)*
*Davis et al., PRL (2020) (Lyon)*

From Pan et al.

resonant-manifold calculation for theoretical steady state
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Resonant manifold: solution of resonance constraints

<table>
<thead>
<tr>
<th>Label</th>
<th>Resonance condition</th>
<th>Solutions</th>
</tr>
</thead>
</table>
| (Ia), (Ib) | \( p = p_1 + p_2 \) \[
\begin{align*}
\frac{k}{m} &= \frac{k_1}{m_1} + \frac{k_2}{m_2 - m_1} \\
\end{align*}
\] | \[
\begin{align*}
m_1^* &= \frac{m}{2k} \left[ k + k_1 \pm k_2 \pm \sqrt{(k + k_1 \pm k_2)^2 + 4kk_1} \right] \\
m_2^* &= m - m_1^* \\
\end{align*}
\] |
| (IIa), (IIb) | \( p_1 = p + p_2 \) \[
\begin{align*}
\frac{k_1}{m} &= \frac{k_1}{m_1} + \frac{k_2}{m_1 - m} \\
\end{align*}
\] | \[
\begin{align*}
m_2^* &= -\frac{m}{2k} \left[ k + k_1 - k_2 - \sqrt{(k + k_1 - k_2)^2 + 4kk_2} \right] \\
m_1^* &= m + m_2^* \\
\end{align*}
\] |
| (IIIa), (IIIb) | \( p_2 = p + p_1 \) \[
\begin{align*}
\frac{k_2}{m_2} &= \frac{k}{m} + \frac{k_1}{m_2 - m} \\
\end{align*}
\] | \[
\begin{align*}
m_1^* &= -\frac{m}{2k} \left[ k - k_1 \mp k_2 - \sqrt{(k - k_1 \mp k_2)^2 + 4kk_1} \right] \\
m_2^* &= m + m_1^* \\
\end{align*}
\] |

Two contributions (local / scale separated): \( \frac{\partial n_p}{\partial t} = I_p \), \( I_p = I_p^{(loc)} + I_p^{(sep)} \)
Stationary state

- WKE with $f = 0$ (non-rotating approximation) is fully scale invariant: can look for general power-law solution (Zakharov-Lvov-Falkovich '92)

$$n(k, m) = Ak^{-a}|m|^{-b}$$

- Conditions for a convergent collision integral require $3 < a < 4, b = 0$ (convergence segment)

- Stationary solution: $a \simeq 3.69, b = 0$
  (scale invariant limit of GM is $a = 4, b = 0$)

Numerical calculation of collision integral along the $b = 0$ segment from Lvov et al., JPO (2011)

$\dot{n}_p = 0$ at $a \simeq 3.7$
A nonequilibrium stationary solution

Stationary balance: look for $\dot{n} = 0$

Power-law ansatz: $n(k, m) = A k^{-a} |m|^{-b}$

Scale invariant limit of GM is $a = 4, b = 0$

$\Rightarrow$ Solution: $a = 3.69, b = 0$

Color map of collision integrand

Projection used for comparison with Pan et al. (2020)

Large contribution from interactions previously neglected: not scale separated, but local, and mainly near horizontal colinearity (Dematteis & Liov, JFM (2021)).

Nonequilibrium state: we can “see” downscale flux, but how to calculate?
Calculation of inter-scale energy fluxes

Method for **energy flux between two “control volumes” in Fourier space**

- **Input volume** $\mathcal{A}$, **output volume** $\mathcal{B}$
- Among all triads, consider only directed energetic links from $\mathcal{A}$ to $\mathcal{B}$ (red arrows); discard all the others (blue arrows)
- Calculate power $\mathcal{P}_{\mathcal{A}\rightarrow\mathcal{B}}$ from **integral conservation equation** in Fourier space
- Wave turbulence analogy to Kraichnan’s ’59 calculation of turbulent fluxes

\[ \mathcal{P}_{\mathcal{A}\rightarrow\mathcal{B}} = \int_{\mathcal{A}} \omega_p T_{p\rightarrow\mathcal{B}} dp \]

*Dematteis and Lvov, arXiv:2205.12899 (Accepted in JFM, 2022)*
Transfer integrals

$P_{\text{out,h}} = \int_{m_{\text{min}}}^{m_{\text{max}}} dm \mathcal{F}_{\text{out,h}}(m)$,

$P_{\text{out,v}} = \int_{\frac{k}{m_{\text{max}}}}^{N \gamma m_{\text{max}}} dk \mathcal{F}_{\text{out,v}}(k)$,

$\left( \begin{array}{c} \mathcal{F}_{\text{out,h}}(m) \\ \mathcal{F}_{\text{out,v}}(k) \end{array} \right) = 4\pi \frac{N^2}{g} (V_0 A)^2 \left( \begin{array}{c} \frac{k_{\text{max}}}{m_{\text{max}}} C_h \\ k_{\text{max}}^{6-2\alpha} m_{\text{max}} C_v \end{array} \right)$,

$C_h = \int_{\frac{f}{N}}^1 ds T_h(s)$,  \hspace{1cm}  $C_v = \int_{\frac{m_{\text{min}}}{m_{\text{max}}}}^{1} ds T_v(s)$,

No more than 10% of downscale flux comes from interactions with large scale separation!
Comparison with finescale parameterization

Total energy flux from waves to turbulence:

\[ \mathcal{P}^{\text{WKE}} = \mathcal{P}_0^{\text{WKE}} f N^{1+\nu} E^2 \left( 1 + O \left( \frac{f}{N} \right)^\nu \right), \quad \nu = 2a + 7 \]

Finescale parameterization: \( \mathcal{P}^{\text{FP}} = \mathcal{P}_0^{\text{FP}} f N^2 E^2 \)

Scaling comparison

GM76: \( a = 4 \Rightarrow \nu = 1 \):
\[ \mathcal{P}^{\text{WKE}} \propto f N^2 E^2 \]

Magnitude comparison

For stationary state we obtain:
\[ \mathcal{P}_0^{\text{WKE}} \approx 9.0 \times 10^{-9} \text{ W/kg} \]

For this solution, finescale formula gives:
\[ \mathcal{P}_0^{\text{FS}} \approx 5.9 \times 10^{-9} \text{ W/kg} \]

Theoretical calculation consistent up to factor 1.5 with empirical value!

Dematteis, Polzin, Lvov, JPO (2022)
Fully rotating problem

(Ongoing work)

Resonant manifold for different test waves (yellow dot) in the $m - \omega$ space
Different energy pathways for different spectra

\[ \epsilon \simeq 7 \times 10^{-10} \text{ W/kg} \]

\[ \epsilon \simeq 2.1 \times 10^{-9} \text{ W/kg} \]
Cross-validation from independent global datasets

Strain estimates from Argo
(provided by Caitlin Whalen)

Our theory with GMACMD input data

- Take regions where both are available (~200 datapoints)
- Compare values (by binning)

[Color map: $10^{-11}$ W/kg (blue) to $10^{-8}$ W/kg (yellow)]
Upper: 250 m – 500 m depth range
Center: 500 m – 1000 m
Lower: 1000 m – 2000 m

# datapoints in each bin under errorbar
(95% confidence level),
main diagonal = perfect agreement,
green = agreement within factor 2,
red = agreement within factor 3
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Open questions with important implications for climate

- **Finescale parameterization**, synthesis of observation, dimensional analysis and intuition: **state-of-the-art** for deep-ocean dissipation and mixing in global models. \( P_0 \) is **determined empirically**

- Physical understanding based on uncontrolled Induced Diffusion approximation: **no clear justification** except that it works!

Summary of main results

- Derivation of formula that descends directly from first principles (via WT closure!), no adjustable parameters
- Consistent scaling; **theoretical** \( P_0 \) agrees with empirical value
- Largest contribution to energy transport is from **local interactions**, neglected in the literature: **new physics to be described**
- New capability to detail intensity and direction of energy fluxes
Open questions with important implications for climate

- **Finescale parameterization**, synthesis of observation, dimensional analysis and intuition: *state-of-the-art* for deep-ocean dissipation and mixing in global models. $P_0$ is determined empirically

- Physical understanding based on uncontrolled Induced Diffusion approximation: *no clear justification* except that it works!

Summary of main results

- Derivation of formula that descends directly from first principles (via WT closure!), no adjustable parameters

- Consistent scaling; *theoretical* $P_0$ agrees with empirical value

- Largest contribution to energy transport is from local interactions, neglected in the literature: *new physics to be described*

- New capability to detail intensity and direction of energy fluxes
Latest breakthrough

Work mainly in collaboration with Arnaud Le Boyer (Scripps) + recent contact with Caitlin Whalen (University of Washington)

- Lifted the “non-rotating assumption” (overcoming significant increase in computational cost)
- Generalized the flux quantification to any input spectral shape
- Creating a new dynamical parameterization able to assimilate spectra from global databases and predict the associated turbulent dissipation and mixing
- Global predictions from current-meter input tested directly against independent finestructure estimates from Argo, showing striking agreement
Next goals

1. Keep increasing confidence in our results by use of other observational programs / microstructure measurements / outputs from high-resolution numerical models.

2. Study seasonal dependence and clarify physics of energy pathways from large-scale sources to dissipation.

3. Search for a general first-principles parameterization with potential to improve representation of internal wave-induced diapycnal mixing in GCMs.

4. Applications to other fields where ocean mixing is crucial (e.g. biological / biogeochemical).
Thank you for your attention!

Collaborators:

Yuri Lvov  
Rensselaer Polytechnic Institute

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Woods Hole Oceanographic Institution

Arnaud Le Boyer  
Scripps Institution of Oceanography

Caitlin Whalen  
University of Washington
Finescale vs microstructure

Microstructure profilers = “real measurements” of turbulence

Direct comparison of finescale estimates (from Argo profiles) and microstructure measurements, for different experiments (black rectangles in mixing map in previous slide)

Whalen et al., JPO (2015)
Simple Boussinesq approximation: neglect density variations when multiplying momentum (Vallis)

\[
D_t \mathbf{v} + \mathbf{f} \times \mathbf{v} = -\frac{\nabla P'}{\rho_0} - \frac{g \rho'}{\rho_0} \hat{k},
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

\[
D_t \rho' + w \partial_z \bar{\rho} = 0,
\]

where \( \mathbf{v} = (u, w) \), and \( \partial_z \bar{\rho} \) is a given stable stratification of the ocean

5 equations in 5 variables
Assumptions

- **Isopycnal coordinates**
- **Horizontal isotropy** and **spatial homogeneity** (constant $N$ in the vertical direction)
- **Zero total potential vorticity**: neglect interaction with large scale vortices
- **Hydrostatic balance** assumption, $\omega \ll N$: $\omega^2 = f^2 + \left(\frac{g}{\rho_0 N m}\right)^2$

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla M}{\rho} = 0, \quad M_{\rho t} + \nabla \cdot (M_{\rho \rho} \mathbf{u}) = 0.
\]

Here $\nabla = (\partial_x, \partial_y)$ is the isopycnal gradient, acting along surfaces of constant density, and $M$ is the Montgomery potential $M = P + g \rho z$.

3 equations in 3 variables
Introduce $u = \nabla \phi(x, \rho, t)$, and $\Pi = \rho M_{\rho \rho} / g$. 
$\Pi$ and $\phi$ are canonically conjugated

$$\Pi_t = \frac{\delta \mathcal{H}}{\delta \phi}, \quad \phi_t = -\frac{\delta \mathcal{H}}{\delta \Pi}.$$ 

with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int \left( \Pi |\nabla \phi|^2 - g \left| \int^{\rho} \frac{\Pi - \Pi_0}{\rho_1} d\rho_1 \right|^2 \right) \, dx \, d\rho.$$ 

Lvov and Tabak 2001 2004

2 (beautifully symmetric) equations in 2 scalar fields
Reduction to equation for the statistics

- $a_p$: normal variables of the linearized internal-wave equation (linear combination of $\phi$ and $\Pi$)

$$a_p := \sqrt{\frac{\omega_p}{2g}} \sqrt{\frac{N}{k}} \Pi_p - i \sqrt{\frac{g}{2\omega}} \sqrt{\frac{k}{N}} \Phi_p$$

- Take a **random Gaussian field** $\sim$ random Gaussian field + **weak nonlinearity assumption**

$$\langle a_p a_p^* \rangle = n_p \delta(p - p')$$

- Reduce from Hamilton’s equations to the Wave Kinetic Equation: “standard step” in Wave Turbulence, but delicate mathematical proof (Simons Collaboration on Wave Turbulence: first rigorous proof, *Deng & Hani, 2021*)
Stationary state

- WKE with \( f = 0 \) (non-rotating approximation) is fully scale invariant: can look for general power-law solution (Zakharov-Lvov-Falkovich '92)

\[
n(k, m) = A k^{-a} |m|^{-b}
\]

- Conditions for a convergent collision integral require \( 3 < a < 4, b = 0 \) (convergence segment)

- **Stationary solution**: \( a \approx 3.69, b = 0 \)
  (scale invariant limit of GM is \( a = 4, b = 0 \))

Numerical calculation of collision integral along the \( b = 0 \) segment from *Lvov et al., JPO (2011)*

\[
\dot{n}_p = 0 \text{ at } a \approx 3.7
\]
Observational evidence of PSI from bi-spectra

MacKinnon et al, JPO (2013)
Bicoherence of the triad \((f, f, 2f)\) at 29°

\[
B(\omega_1, \omega_2) = E[X_{\omega_1}^* X_{\omega_2}^* X_{\omega_1+\omega_2}]
\]

\[
b^2(\omega_1, \omega_2) = \frac{|B(\omega_1, \omega_2)|^2}{E[|X_{\omega_1} X_{\omega_2}|^2]E[|X_{\omega_1+\omega_2}|^2]},
\]

Experiment conducted North of Hawaii where \(M_2\) tide is very strong
Evidence of ID (and more) in high-resolution model

Pan et al, JPO (2020)

\[ \frac{\partial n_p}{\partial t} = \mathcal{J} \sum_{p=p_1+p_2} V(p, p_1, p_2) \langle a_p^* a_{p_1} a_{p_2} \rangle - 2 \mathcal{J} \sum_{p_1+p_2} V(p_1, p, p_2) \langle a_{p_1}^* a_p a_{p_2} \rangle \]

From Pan et al.

Evaluation of “collision integrand” from direct numerical simulations of a high-resolution numerical model (∼ internal-wave resolving) forced temporally and at boundaries by a GCM reference run.

resonant-manifold calculation for theoretical steady state
PSI and other resonant triads in water tanks

Grayson et al. (2022) (Cambridge)

Size contraints: still unclear whether a proper “continuum” can be created in the lab. But these are exciting new developments

Davis et al., PRL (2020) (Lyon)