Solutions of wave kinetic equations with constant fluxes

1. Conservation laws and fluxes

- Macroscopic conservation laws are generically associated to microscopic conservations
  (cf Bredon - Desvillettes for counterexamples)

- They are associated to some symmetries of the collision operator, more visible in the weak formulation

- Using this weak formulation, it is straightforward to define fluxes introducing derivatives via Taylor's formula

$\boxed{\text{3 wave equation}}$

$$\int Q_{3w}(n,n)\,\varphi\,dk = \int dk\,dk_1\,dk_2\;\sigma_{kk_1k_2}\,\delta_{k-k_2-k_1}\,\delta_{\Omega^k_{k_1k_2}}\; n\,n_1\,n_2 \left(\frac{1}{n}-\frac{1}{n_1}-\frac{1}{n_2}\right)\left(\varphi-\varphi_1-\varphi_2\right)$$

→ collision invariants : k, ω_k

$$\int Q_{3w}(n,n)\,k_i\,dk = \int Q_{3w}(n,n)\,\omega_k\,dk = 0$$

⚠ no mass conservation!

→ isotropic case $\begin{cases} \omega_k = \Omega(|k|) \\ n = n(|k|) \end{cases}$

$$\Sigma_{kk_1k_2} = |k|^{d-1}|k_1|^{d-1}|k_2|^{d-1}\int \sigma_{kk_1k_2}\,\delta_{k-k_1-k_2}\,d\theta_k\,d\theta_{k_1}\,d\theta_{k_2}$$

energy flux $\int Q_{3w}(n,n)\,\omega\,\psi\,dk = -\int \nabla_k \cdot J^{E}_{3w}(n,n)\;\psi\,dk$

$$= \int J^{E}_{3w}(n,n) \cdot \nabla_k \psi\,dk$$

If $\varphi = \omega\psi$ then $\varphi - \varphi_1 - \varphi_2 = (\omega_1 + \omega_2)\psi - \omega_1\psi_1 - \omega_2\psi_2$

$$= \omega_1(\psi - \psi_1) + \omega_2(\psi - \psi_2)$$

$$= \omega_1 \int_{k_1}^{k} \psi' + \omega_2 \int_{k_2}^{k} \psi'$$

By Fubini, one can choose

$$|k|^d J^{E}_{3w}(n,n) = 2k \int dk'\,dk_1\,dk_2 \sum_{k' k_1 k_2} \mathbb{1}_{k_1 \leq k \leq k'}\, \delta_{\Omega^{k'}_{k_1 k_2}}\, n' n_1 n_2 \left(\frac{1}{n'} - \frac{1}{n_1} - \frac{1}{n_2}\right)\omega_1$$

$\boxed{\text{4-wave equation}}$

$$\int Q_{4w}(n,n,n)\,\varphi\,dk = \frac{1}{4}\int dk\,dk_1\,dk_2\,dk_3\,\delta_{k+k_2-k_1-k_3}\,\delta_{\Omega^{kk_2}_{k_1k_3}}\,n n_1 n_2 n_3 \left(\frac{1}{n}+\frac{1}{n_2}-\frac{1}{n_1}-\frac{1}{n_3}\right)\left(\varphi+\varphi_2-\varphi_1-\varphi_3\right)$$

→ collision invariants : $1, k, \omega_k$

→ mass flux $$J^M_{4w}(n,n,n) = \frac{k}{2|k|^d}\int dk'\,dk_1\,dk_2\,dk_3\,\Sigma_{k'k_1k_2k_3}\,\mathbb{1}_{k\in[k_1,k']}\,\delta_{\Omega^{k'k_2}_{k_1k_3}}\,n'n_1n_2n_3\left(\frac{1}{n'}+\frac{1}{n_2}-\frac{1}{n_1}-\frac{1}{n_3}\right)$$

→ energy flux $$J^E_{4w}(n,n,n) = \frac{k}{4|k|^d}\int dk'\,dk_1\,dk_2\,dk_3\,\Sigma_{k'k_1k_2k_3}\,\delta_{\Omega^{k'k_2}_{k_1k_3}}\,n'n_1n_2n_3\left(\frac{1}{n'}+\frac{1}{n_2}-\frac{1}{n_1}-\frac{1}{n_3}\right)\left[\omega_1\mathbb{1}_{k\in[k_1,k']}+\omega_3\mathbb{1}_{k\in[k_3,k']}-\omega_2\mathbb{1}_{k\in[k_2,k']}\right]$$

Non isotropic fluxes

- In the radial case, the flux is <u>defined uniquely</u> (up to a constant) by its divergence.

- In the non radial case, we need to prescribe a <u>gauge condition</u>

Assume that w_k is isotropic, and that σ is invariant by rotation. One can design a procedure (Escobedo, Golse, SR) to define uniquely the flux J^ρ for any radial collision invariant ρ.

$\longrightarrow$ Is it the only flux J such that for any isometry R

$$R^t J(n)(Rk) = J(n \circ R)(k) \ ??$$

2. Constant flux distributions

- When forcing and dissipation occur at different scales, stationary solutions should exhibit a cascade mechanism compatible with conservation laws.

- This cascade is classically described by a local equation obtained by Zakharov's transform

- We propose here a more systematic approach, which does not require a strong intuition of the physical mechanism.

Solutions of the divergence equation

Proposition: If the flux $J^\rho(n)(k) = k\, j^\rho(n)(k)$ with $j(n)$ radial and homogeneous of degree $-d$, then

$$\rho Q(n) = \operatorname{div} J^\rho(n) = c\,\delta_0 \quad \text{in } \mathcal{D}'$$

where c is the residue of $j^\rho(n)$ at 0.

Proof: by Euler's relation for homogeneous functions

$$k \cdot \nabla j(n)(|k|) = -d\, j(n)(|k|) \quad \text{for } k \neq 0$$

Therefore

$$\operatorname{div} J(n)_{(k)} = k \cdot \nabla j(n)_{(k)} + d\, j(n)_{(k)} = 0 \quad \text{for } k \neq 0$$

The distribution div $J(n)$ is supported on $\{0\}$ and homogeneous of degree $-d$.

$$\operatorname{div} J(n) = c\,\delta_0 \quad \text{in } \mathcal{D}' \qquad \blacksquare$$

Remarks: this basic result provides a very general setting to construct Kolmogorov-Zakharov (power-law) spectra provided that:

* the dispersion relation ω_k is isotropic and homogeneous
* the cross-section σ is homogeneous.

However there are a few technical issues.

Non interacting condensate regime (cf Escobedo-Velazquez)

- In the previous setting, we implicitly consider that the part of the system with $k=0$ does not interact with the rest of the system
  → instantaneously absorbed by an infinite sink.

- This means that in the weak formulation of the equation the domain of integration does not include 0.

Locality (Balk-Zakharov, Collot-Dietert-Germain)

In order that the equation makes sense, the integrals defining the collision operator and the flux have to be convergent

Proposition (Collot-Dietert-Germain) For the 4-wave kinetic equation with $\omega_k = |k|^2$ and $\sigma \equiv 1$ a distribution n such that $n(k) \underset{0}{\propto} |k|^{-\alpha}$ and $n(k) \underset{+\infty}{\propto} |k|^{-\beta}$

- is local if $\beta > 1$ and $\alpha \in [0, \frac{5}{4}[$
- is weakly local if $\beta > 1$ and $\alpha \in [0, \frac{3}{2}[$

Direction of the cascade

- The sign of the residue indicates whether 0 is a source (+) or a sink (-).

- For the 4-wave equation, the Fjortoft argument shows that the energy must have a direct cascade (positive flux at 0) the mass must have an indirect cascade (negative flux at 0).

- When the flux has the wrong sign, one expects warm cascades (cf Proment-Onorato-Asinari-Nazarenko)

3. Self similar profiles

- The Kolmogorov-Zakharov solutions n^{KZ} are expected to predict the long time behavior of realistic physical systems in some specific range of wavenumbers.

- This behavior depends on the capacity of the KZ solution defined as

$$\begin{cases} \int_1^\infty n_{KZ} & \text{for direct cascades} \\ \int_0^1 n_{KZ} & \text{for indirect cascades} \end{cases}$$

Evolution scenarios (cf Falkovich, Shafarenko)

- In the infinite capacity case, with forcing and dissipation, the Kolmogorov spectrum should form behind some relaxation front $k_{front} \sim t^{1/h}$

- In the finite capacity case, with forcing and dissipation, the solution should grow as a whole in the inertial range, but not in a self similar way.

(The regime without forcing is maybe simpler)

Analysis of the coagulation equation

$$Q_c(n,n) = \frac{1}{2}\iint \sigma(k_1,k_2)\, n_1 n_2\, \delta_{k-k_1-k_2} - n\int_0^\infty \sigma(k,k_1)\, n_1$$

with $\sigma(\lambda k_1, \lambda k_2) = \lambda^\gamma \sigma(k_1,k_2)$ $(\gamma<1)$

- similar to the 3-wave equation with collision invariant k

- direct cascade (coagulation) with infinite capacity

$$\phi_{KZ}(k) \sim |k|^{-(3+\gamma)/2}$$

$\Rightarrow$ Find a self-similar profile $n_s(t,k) = t^{-\alpha}\phi_s(t^{-\beta}k)$ describing the asymptotic behavior of $n(t,k) = t^{-\alpha}\Phi(\log t, t^{-\beta}k)$

- Scaling relations

$$\begin{cases} \int n(t,k)\,dk = t^{2\beta-\alpha} \int \Phi(\log t, \xi)\,\xi\, d\xi = O(t) \\ \partial_t n = -t^{-\alpha-1}\left(\alpha \Phi + \beta \xi \partial_\xi \Phi - \partial_\tau \Phi\right) = O\left(t^{\beta-2\alpha+\beta\gamma}\right) \end{cases}$$

$$\Longrightarrow \quad \beta = \frac{2}{1-\gamma}, \quad \alpha = \frac{3+\gamma}{1-\gamma}$$

- Profile equation

$$\alpha \Phi_s + \beta \xi \partial_\xi \Phi_s + \mathcal{L}_c(\Phi_s, \Phi_s) = 0 \qquad (*)$$

- Matching conditions

$\xi \to 0$ (behind the front) $\quad \Phi_s(\xi) \sim \xi^{-(3+\gamma)/2}$

$\xi \to \infty \quad \Phi_s(\xi)$ exponentially small.

Theorem (Ferreira, Franco, Velazquez) : under technical assumptions on σ, there exists a self similar profile

- with finite mass $\int \xi \Phi_s(\xi)\, d\xi$
- and bounded flux $\int \sigma(\xi_1, \xi_2)\, \Phi_{s1} \Phi_{s2}\, \xi_2\, 1_{\xi_2 \leq \xi \leq \xi_1 + \xi_2}$

satisfying $(\star)$ in the sense of distributions.

Furthermore, $\displaystyle \frac{1}{z} \int_{bz}^{z} \Phi_s(\xi)\, d\xi \underset{z \to 0}{\sim} C z^{-(3+\gamma)/2}$

$\displaystyle \limsup_{\xi \to \infty} \Phi_s(\xi) \exp(C\xi) < +\infty$.

Open questions
- uniqueness of Φ_s and behavior at $\xi = 0$.
- basin of attraction of Φ_s

Analogies with type I and type II blow up ?

Is it possible to rephrase the property of finite capacity in terms of solutions of $(*)$?

cascade to $+\infty$	?? $\longleftrightarrow$ blow up at 0
self similar behavior at 0	self similar behavior at ∞.
exponential decay at ∞	smooth in 0
$\hookrightarrow$ infinite capacity	$\hookrightarrow$ type I
backward flux ??	ground state behavior
$\hookrightarrow$ finite capacity	$\hookrightarrow$ type II