Spectral-Element Magnetohydrodynamics for Stellarators*

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Simons Collaboration on Hidden Symmetries and Fusion Energy
New York          March 21-22, 2024

*Work supported by DOE OFES through DE-SC0018642, DE-FG02-99ER54546, and DE-SC0024548.
Introduction: Plasma-\( \beta \) is important for fusion.

- Fusion power balance amounts to a relation between the output/input power ratio \( Q \), temperature \( T \), and the \( nT\tau_E \) “triple product.”
- Plasma-\( \beta \) appears when rewriting pressure:

\[
nk_B T\tau_E = \beta_i \left( \frac{B^2}{2\mu_0} \right) \tau_E
\]

Which represents confinement efficiency, transport physics, engineering & manufacturing costs, and costs.

Contours of scientific-\( Q \) for D-T fusion; \( Q = 1 \) is scientific breakeven.
Stellarator $\beta$-values rivaling tokamaks have been achieved.

- LHD achieved $\beta = 4\%$ at large aspect ratio above linear ideal-MHD stability limits [Watanabe, et al., NF 45, 1247 (2005)].

- Comparing high-$\beta$ operation in W7-AS (low shear) and LHD (high shear), Weller, et al. note that the Shafranov shift and heating power are more relevant than linear stability [NF 49, 065016 (2009)].

- W7-X operation with island divertor achieves central $\beta \cong 4\%$ [Sunn Pederson, et al., NF 62, 042022 (2022)].
High-$\beta$ effects are studied with relaxation and time-dependent computations.

HINT2 computation of LHD topology and profile relaxation for LHD [Y. Suzuki, et al., IAEA 2008 TH/P9-19].

PIES computation of W7-AS force balance at $\beta = 2.5\%$ [A. Reiman, et al., NF 47, 572 (2007)].

NIMROD MHD evolution of a torsatron subject to heating [T. Bechtel, PhD Dissertation, Univ. WI-Madison, 2021].

M3D-C1-S MHD evolution from W7-AS VMEC result [Y. Zhou, NF 61, 086015 (2021)].
Time-dependent nonlinear computations have advantages.

- Physical time provides a meaningful connection among states.
  - Prescriptions and constraints for relaxation are not required.
- Instabilities and their consequences are modeled when they arise.
- Magnetic topology evolution is described.
- Physical effects beyond base MHD models may be added.
- Externally generated states may be used:
  - We may perturb about the state if it is an accurate force balance.
  - Alternatively, the state may be used as an initial condition.
Perturbing about externally generated states is precarious.

Constraint of \( X = 0 \) is consistent with force balance, \(-\partial E/\partial X = 0\), without the constraint.

Constrained configuration is not consistent with force balance after removing the \( X = 0 \) constraint.

A constrained force balance computation would not distinguish the two, possibly apart from residual error.
For nonlinear time-dependent computations, non-ideal magnetohydrodynamics is the base model.

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V}) = -\nabla \cdot \mathbf{\Gamma}_a
\]

particle continuity with artificial diffusive flux

\[
m n_i \left( \frac{\partial}{\partial t} \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \nabla \cdot \left[ \frac{1}{\mu_0} \mathbf{BB} + \Pi - \left( P + \frac{B^2}{2\mu_0} \right) \mathbf{I} \right] + \mathbf{S}_v
\]

momentum density; \( n_i = n/Z \);

\[ P = (1 + 1/Z)nT \quad \text{for single-}T \]

temperature evolution

\[
\frac{n_i}{(\gamma - 1)} \left( \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right) = -n_i T \nabla \cdot \mathbf{V} - \nabla \cdot \mathbf{q}_i + S_{Qi}
\]

temperatue evolution

\[
\mathbf{\sigma} \cdot \left( \frac{\partial}{\partial t} \mathbf{A} + \nabla \chi \right) = \mathbf{\sigma} \cdot \mathbf{V} \times \mathbf{B} - \nabla \times \nabla \times \mathbf{A} \quad \mathbf{B} = \nabla \times \mathbf{A}
\]

resistive-MHD Ohm’s law & vector potential

or

\[
\frac{\partial}{\partial t} \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) + \kappa_B \nabla \nabla \cdot \mathbf{B}
\]

combined Faraday / Ohm’s laws with error correction
Closures approximate plasma transport; source terms may include numerical corrections.

- Thermal conduction and viscous stress are anisotropic.
  - $q = -n \left[ (\chi_\parallel - \chi_{iso}) \hat{b} \hat{b} + \chi_{iso} I \right] \cdot \nabla T$. \hspace{1em} $\hat{b} = B |B|^{-1}$
  - $\Pi = \nu_\parallel m n_i (I - 3 \hat{b} \hat{b}) \hat{b} \cdot \hat{b} - \chi_{iso} W$ \hspace{1em} $W = \nabla V + \nabla V^T - \frac{2}{3} \mathbf{I} \cdot \nabla \cdot V$

- Equations include numerical error-correcting terms.
  - Diffusive particle flux: $\Gamma_a = -D_n \nabla n + D_h \nabla \nabla^2 n$
  - Momentum correction: $S_v = m \nabla \cdot \Gamma_a$
  - Ohmic and viscous heating + energy correction:
    $$S_{Q_i} = f_i [\eta J^2 - \Pi : \nabla V] - \left[ \frac{f_i m V^2}{2} - \frac{T}{Z^{(\gamma - 1)}} \right] \nabla \cdot \Gamma_a$$
MFE conditions of interest allow one simplification.

- Shock propagation is not a concern.
- Temporal scales are separated.
- Geometry is nontrivial.
- There is a divergence constraint on $\mathbf{B}$.
- Transport is anisotropic.
- Localized features arise.
- Successful simulation depends on distinguishing different modes.

Ideal-MHD eigenvalues in a straight cylinder,
$\mathbf{B}_0 = B_0\hat{z}$, $\gamma = 1$, $\beta = 1$, $m = 1$, $k_z = 2$. 
Unstable modes tend to tap free energy near spatial resonances.

- Modes are driven by \( \nabla P_0 \parallel \hat{b} \cdot \nabla \hat{b} \) and by \( \nabla (J_{\parallel 0} / B_0) \).
- Localization implies that numerical results depend on limit of resolution.

Eigenmode for ideally unstable interchange.
The compressible slab $g$-mode indicates the subtleties.

- Ideal-MHD eigenmodes satisfy

$$\frac{\partial}{\partial y} \left[ \left( k^2 S - \frac{\rho^2 \omega^4}{\rho} \right) \frac{\partial}{\partial y} \xi_y - \rho \omega^2 J_{z0} B_{x0} A \xi_y \right] - \left( A + \frac{\rho g}{L} \right) \xi_y = 0$$

$$A = \rho \omega^2 - F^2 ; \quad F = k \cdot B_0 ; \quad S = \rho \omega^2 (B_0^2 + \gamma P_0) - F^2 \gamma P_0$$

subject to $\xi_y \to 0, y \to \pm \infty$.

- Terms from compression at fine scale
- Terms from field-line bending at fine scale

- Reduced systems satisfy

$$\frac{\partial}{\partial y} \left[ (\omega^2 - y^2) \frac{\partial}{\partial y} \phi \right] - D \phi = 0$$

$D > 1/4$ allows instability, $\omega^2 < 0$. 

\[ J_{z0} B_{x0} = \rho_0 \left( 1 + \frac{y}{L} \right) g \]

slab equilibrium configuration
Methods: NIMSTELL is a stellarator variant of NIMROD*.

- Both use a plane of 2D elements with finite Fourier series for a periodic coordinate.
- FE nodes are based on Legendre polynomials, allowing either \( h \) or \( p \) refinement.
- NIMSTELL’s geometry is expanded as a Fourier series in a generalized toroidal angle.

\[
R(\xi, \eta, \zeta) = R_0(\xi, \eta) + \sum_{n=1}^{N} [R_n(\xi, \eta)e^{in\zeta} + cc.] \\
Z(\xi, \eta, \zeta) = Z_0(\xi, \eta) + \sum_{n=1}^{N} [Z_n(\xi, \eta)e^{in\zeta} + cc.] \\
\phi(\xi, \eta, \zeta) = \phi_0(\xi, \eta) + \sum_{n=1}^{N} [\phi_n(\xi, \eta)e^{in\zeta} + cc.] + \zeta
\]

Mesh and \( T \) contours from anisotropic conduction in WISTELL-A configuration.

Vector potential is expanded in the $H$(curl) space.

- Nédélec edge elements* form a basis for the $H$(curl) function space.

- The $\sigma_k(x), x \in \{\xi, \eta\}$ are discontinuous.
- The $\lambda_k(x), x \in \{\xi, \eta\}$ are continuous expansions and 1 degree higher.

Magnetic field satisfies expected constraints.

\[ A = \sum_{k,l} [A_{\xi_{k,l}}(\zeta) \sigma_k(\zeta) \lambda_l(\eta) \nabla \xi + A_{\eta_{k,l}}(\zeta) \lambda_k(\xi) \sigma_l(\eta) \nabla \eta + A_{\zeta_{k,l}}(\zeta) \lambda_k(\xi) \lambda_l(\eta) \nabla \zeta] \]

- Divergence constraint \( \nabla \cdot B = 0 \) within elements follows from \( B = \nabla \times A \).
- Continuity of \( B \cdot \hat{n} \) across element boundaries is inherent from the formulation.
  - For example, consider \( \hat{n} = \frac{\nabla \xi}{|\nabla \xi|} \propto \frac{\partial R}{\partial \eta} \times \frac{\partial R}{\partial \zeta} \) element surface normal:
    - \( \frac{\partial R}{\partial \eta} \) and \( \frac{\partial R}{\partial \zeta} \) are continuous by construction.
    - Continuous by Fourier expansion

\[ \hat{n} \cdot \nabla \times A = \frac{\nabla \xi}{|\nabla \xi|} \cdot \left[ \sum_{k,l} [A_{\zeta_{k,l}}(\zeta) \lambda_k(\xi) \lambda_l'(\eta) - \lambda_k(\xi) \sigma_l(\eta) A'_{\eta_{k,l}}(\zeta)] \right] \frac{1}{j} \frac{\partial R}{\partial \xi} \]

and \( \frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{1}{j} \frac{\partial R}{\partial \xi} = \left| \frac{\partial R}{\partial \eta} \times \frac{\partial R}{\partial \zeta} \right|^{-1} \lambda_l'(\eta) \) and \( \sigma_l(\eta) \) are continuous for \( 0 < \eta < 1 \)
1D cylindrical ideal-MHD eigenvalue results compare formulations.

- $H(\text{curl})$ has continuous $rA_\theta$ and $A_z$ over $r$, discontinuous $A_r$ of 1 degree lower.
- Results here are for the basic uniform-$B_0$ profile.

Numerical eigenvalues with $A_r$ of the same degree as $V$ and $p$.

Numerical eigenvalues with $rA_\theta$ and $A_z$ of the same degree as $V$ and $p$.

Labeling
- W: Weyl, $\chi = 0$
- d: damping, $\nabla^2 \chi = d_A \nabla \cdot A$
- D: diffusive, $\chi = -D_A \nabla \cdot A$
- L2: diffusive with $\chi$ in $L2$, $A$ in $H1$
- pt: total, not plasma, pressure

All formulations with $A$ have $\omega = 0$ modes; Weyl has the most.
- Having $\chi$ in $L2$, hence $A$ in $H1$, misses the Alfvén spectrum.
Eigenvalue results on interchange consider fidelity to fine-scale response.

- Equilibrium profile\(^*\) has \(B_{z0} = 1, q(r) = (1 + c_2^2 r^2)/Rc_1,\)
  \(D_s(r) = -\frac{1}{rB_{z0}} \left(\frac{q}{q'}\right)^2 P_0' = c_1^2 [(1 + c_2^2 r^2)c_2^4 r^2]^{-1}\)
- Conditions with \(c_1 = 4/7, c_2 = 10/7\) are stable for \(r_s > 0.466\).
- Computations with \(m = 4, k = -1.784\) have \(D_s = 0.443,\) cubic \(V\) and \(p,\) and the damping formulation.
- Mesh is uniform in \(r.\)


Convergence “from the stable side” is important.

Eigenvector for the 80-element computation displays localization.
NIMSTELL reproduces the resistive version of the mode.

- NIMSTELL FE mesh is 2D, but the formulation is based on the 1D eigenvalue results.

Result on $V_z$ for $\eta = \chi_{iso} = \nu_{iso} = 10^{-7}$ with 48x24 mesh of biquartic elements, $n=0,1$.

Twisted mesh case has $\eta = \chi_{iso} = \nu_{iso} = 10^{-6}$ has $0 \leq n \leq 5$ without aligning the twist with the helical mode.
For a given mesh, we find that a minimum level of resistivity is needed.

• A series of computations uses biquartic elements without packing near the rational surface.
• Mesh sizes ranged from $24 \times 16$ to $48 \times 24$ to $96 \times 48$, all having the central region with a non-polar arrangement.
• Unstable noise develops if resistivity is too small for a given resolution.
  • $24 \times 16$ needs $\eta \sim 10^{-6}$.
  • $48 \times 24$ needs $\eta \sim 10^{-7}$.
• Likely related to $\nabla \times \nabla \times \mathbf{A}$ providing coercivity in the $\mathbf{A}$-advance—not evident from 1D eigenmode results, apart from $\omega = 0$ modes.

The lowest-resolution mesh is effectively $24 \times 16$. Element curvature is not shown.
We are also verifying current-driven tearing.

- Cylindrical tearing in a pinch profile is a standard NIMROD verification case.
- Toroidal tearing in a twisting elliptical geometry required opposite helicity.
- DESC* was used to create the $N_p = 1$, $e = 0.8$ equilibrium.

Contours of $V_z$ for an $S = 10^6$, $Pm = 10^{-3}$ computation

Contours of $V_\phi$ from a computation with $S = 2 \times 10^4$, $Pm = 2 \times 10^{-3}$.

Published JOREK results facilitate toroidal verification.

- The configuration has $N_p = 5$, $e = 0.76$.
- We used DESC to generate the equilibrium for NIMSTELL.
- The (2,1) mode requires $0 \leq n \leq 10$ for minimal toroidal resolution.

Comparison of tearing growth rates for W7-A case over a scan of resistivity values.
Coercivity is again a concern at small resistivity values.

✗ Distorting toroidal meshes lead to unphysical instability with the $A$ representation — not a resolution issue.

- Twisting meshes avoid the numerical instability.
- Computations have $e = 0.8$, $N_p = 1$, $S = 1 \times 10^5$, $Pm = 0.1$.
- Tearing mode at low-$n$ (left, from twisted mesh) is expected.
- Ballooning-like mode (right, distorting mesh) is not.

Overlay of unwrapped tori compares twisting (black) and distorting (red) meshes.
Slab geometry reproduces the challenge for $A$ in $H(\text{curl})$ at large $S$.

- Numerical error is reproduced with a mesh distorted over $\zeta$.
- Parameters have $S = 10^6$, $Pm = 0.5$.
- Tearing mode is symmetric over $\zeta$.
- Mesh distortion is $n = 1$; computations have $0 \leq n \leq 5 +$ with dealiasing.
- Coercivity of $A$-equation at small $\eta$-values may be the issue.

Slab-geometry computation with $B$ in $H1$ reproduces symmetric tearing mode. $[V_x$ is shown.]

Slab-geometry computation with $A$ in $H(\text{curl})$ produces numerical mode.
Accurate anisotropic transport is also important.

- Thermal transport computations solve
  \[ \nabla \cdot \left[ (\kappa_\parallel - \kappa_\perp) \mathbf{b} \mathbf{b} \cdot \nabla T + \kappa_\perp \nabla T \right] + Q = 0 . \]

- Straight configuration has
  \[ \mathbf{B} = \nabla \left[ C I_t \left( \frac{N_p r}{R} \right) \cos \left( l \theta - \frac{N_p z}{R} \right) + z \right] \]
  with \( l = 2 \), \( R/N_p = 4 \).

- The separatrix has \( \chi \)-points at
  \( r = 3.46 \), which is inside the domain.

- Computations have \( \kappa_\parallel / \kappa_\perp = 10^6 \).

Twisting mesh and contours of temperature. Mesh is \( n = 0,1 \) only.

Distorting mesh and contours of temperature. Mesh has a broad \( n \) spectrum.
Results support meshing by symmetry, where possible.

Twisting mesh follows symmetry and converges readily. \( p \)-refinement uses 960 elements (32 azimuthally); \( h \)-refinement uses bicubics.

Distorting mesh needs convergence with respect to Fourier expansion, \( 0 \leq n \leq N \). 1340 biquintic elements are used.
There are commonalities and differences with other efforts.

- The stellarator version of M3D-C1 [Y. Zhou, NF 61, 086015 (2021)] uses Bell 2D (reduced quintic) elements and Hermite cubics for a split-implicit potential representation of full MHD.
- The stellarator version of JOREK [N. Nikulsin, et al., PoP 29, 063901 (2022)] uses 2D G1 Bezier elements and 1D Fourier for implicit reduced MHD.
- The NIMSTELL time advance is semi-implicit.

Thrusts for the UW-Madison M3D-C1 effort also include design validation. [Courtesy of A. Wright]
NIMSTELL directions

• Implementation of remaining nonlinear terms and nonlinear applications
• Continue numerical investigation of vector-potential formulation
• Benchmark & apply new $H1$ representation of $B$
  • Boundary condition equation needs preconditioning
  • Provides an alternative representation
• Enhance Fourier-banded preconditioner for quasi-symmetric configurations
• Explore mesh optimization, as allowed by the generalized toroidal angle

Meshes over one FE-Fourier plane. Lower panel is optimized to reduce variation in $J$ along constant-$\eta$ gridlines.
Conclusions

• While expanding $A$ in $H(\text{curl})$ respects the physical divergence constraint on $B$, stability at small $\eta$-values is a numerical challenge.

• Having an alternative representation for $B$ will facilitate applications while the study of vector potential continues.

• Mesh alignment with $B$ is not required for accurate anisotropic transport, but aligning to symmetries helps.

• Using externally generated states as initial conditions is safest.