Quantum Chern-Simons Theory and Resurgence

by

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Recall the Witten-Reshetikhin-Turaev Quantum Invariant:

\[ Z^{(r)}(M, L, \lambda) \in \mathbb{C} \]

- \( r \in \{2, 3, \ldots\} \).
- \( M \) closed oriented 3-manifold
- \( L \subset M \) embedded link, each component colored by representations \( \lambda \) of \( SU(2) \)

These invariants \( Z^{(r)} \) are precise mathematical construction of Witten’s \( SU(2) \) quantum Chern-Simons theory \( (r = k + 2) \):

\[
Z^{(r)}(M, L, \lambda) \overset{\text{def}}{=} \int_{A \in \mathcal{A}_{SU(2)}/G_{SU(2)}} \exp(2\pi ik \text{CS}(A)) \text{tr} \lambda(\text{Hol}_L(A)) \, DA
\]

where

- \( \mathcal{A}_{SU(2)} \) space of all connections in a principal \( SU(2) \) bundle over \( M \)
- \( G_{SU(2)} \) group of gauge transformations in the principal bundle over \( M \)
- \( \text{Hol}_L(A) \) holonomy around \( L \) w.r.t. \( A \in \mathcal{A}_{SU(2)} \).
Recall the Witten-Reshetikhin-Turaev Quantum Invariant:

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\[ \int \ldots DA \] Still no known mathematical definition of Feynman path integrals!
Quantum Chern-Simons theory

$Z^{(r)}$ 2 + 1 dimensional Topological Quantum Field Theory:

- 2 dim. part of $Z^{(r)}$: Modular functor
  - $\Lambda_r$ finite label set for $Z^{(r)}$ with involution $\dagger : \Lambda_r \to \Lambda_r$

$$Z^{(r)} : \left\{ \text{Category of closed oriented surfaces with } \Lambda_r\text{-labeled marked points} \right\} \to \left\{ \text{Category of finite dim vector spaces over } \mathbb{C} \right\}$$

  - $Z^{(r)}(\tilde{\Sigma}, \lambda_1, \ldots, \lambda_n) \cong Z^{(r)}(\Sigma, \lambda_1^\dagger, \ldots, \lambda_n^\dagger)^*$
  - $Z^{(r)}(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda_r} Z^{(r)}(\Sigma, \mu, \mu^\dagger, \lambda)$

  $\tilde{\Sigma} = \Sigma$ with reversed orientation.
  $\Sigma_c$ glue of blow up of $\Sigma$ at two parked points.

- 3 dim. part of $Z^{(r)}$:

  $X$ : compact 3–manifold
  $(L, \partial L) \subseteq (X, \partial X)$ : oriented link
  $\lambda : \Lambda_r$-labeling of $L$

$$Z^{(r)}(X, L, \lambda) \in Z^{(r)}(\partial X, \partial L, \partial \lambda)$$

- $Z^{(r)}$ satisfies the Atiyah-Segal-Witten T.Q.F.T. axioms:

$$Z^{(r)}(X_1, L_1, \lambda_1) \circ Z^{(r)}(X_2, L_2, \lambda_2) = Z^{(r)}(X_1 \cup_{\partial} X_2, L_1 \cup_{\partial} L_2, \lambda_1 \cup_{\partial} \lambda_2)$$

2 dim. Modular Functor part determines entire TQFT.
The three different approaches to the WRT-TQFT (Conjectured by Witten):

- Modular Tensor Categories
  - Reshetikhin-Turaev
  - The modular Functor $Z_{MTC}^{(r)}$
  - A.-Ueno

- Conformal Field Theory
  - Tsuchiya-Ueno-Yamada
  - A.-Ueno
  - The modular Functor $Z_{CFT}^{(r)}$
  - A.-Henke

- Geometric Quantization of Moduli Spaces of flat connections
  - Witten, Hitchin, Thaddeus, Jeffrey, ... Wentworth, Szenes, Ramadas, ...
  - A.-Bjerre, A.-Henke, ...
  - The modular Functor $Z_{MS}^{(r)}$
  - Beauville, Sorger, Pauly, ... Dim. agree Laszlo (g>2) Iso of MCG rep.
  - A.-Henke (g=0) Iso of MCG rep.
Categorification

- Khovanov type Homologies
- Categorifying Link invariants
- Rest???

Decat.

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  - A.-Ueno

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Resurgence in TQFT: In a number of different publications Witten, Garoufalidis, Gukov-Marino-Putrov and Gukov-Pei-Putrov-Vafa, Garoufalidis-Gu-Marino-Wheller have proposed/argued one can in lots of examples apply the theory of Resurgence to the partition function of SU(2) Chern-Simons theory \( r = k + 2 \)

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\mathcal{Z}^{(r)}(M) = \int_{A \in \mathcal{A}_{SU(2)}/\mathcal{G}_{SU(2)}} \exp(2\pi i k \text{CS}(A)) \, DA
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by thinking of

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as a real "middle dimensional cycle" and decomposing it into

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in this path integral setting.
The Reshetikhin-Turaev formula for the colored Jones polynomial via Quantum R-matrices

Recall $r \in \{2, 3, \ldots \}$. Quantum integers are $[l] = \frac{\sin(l \frac{\pi}{r})}{\sin(\frac{l}{r})}$ and $[l]! = \prod_{i=1}^{l}[i]$.

Let $\mathcal{L} = \frac{1}{2}\mathbb{Z}_+ \cup \{0\}$ and $\Lambda_r = \{\lambda \in \mathcal{L} \mid 0 < 2\lambda + 1 < r\}$.

For $\lambda \in \Lambda_r$ let $V^{(\lambda)}$ be a complex v.s. of dim. $2\lambda + 1$, basis $e^{(\lambda)}_i$ indexed by

$$i \in B_\lambda = \{-\lambda, -\lambda + 1, \ldots, \lambda\}.$$

For $\nu, \mu \in \mathcal{L}$ consider the braiding isomorphism

$$C^\pm(\nu, \mu) : V^{(\nu)} \otimes V^{(\mu)} \to V^{(\mu)} \otimes V^{(\nu)},$$

Given by

$$C^\pm(\nu, \mu) \left( e^{(\nu)}_i \otimes e^{(\mu)}_j \right) = \sum_{v, w} C(\nu, \mu)_{i, j; v, w} e^{(\mu)}_v \otimes e^{(\nu)}_w,$$

$$C^+(\nu, \mu)_{i, j; v, w} = \delta^v_i \delta^w_j \frac{[\nu + w]! [\mu - v]!}{[\nu + i]! [\nu + j]! [\mu - v]! [\mu - j]!} e^{i \pi} (4ij - 2(w-i)(i-j) - (w-i)(w-i+1))$$

$$C^-: \quad \begin{array}{c} w \geq i, \\ j \geq v \end{array}$$

$$C^-(\nu, \mu)_{i, j; v, w} = \delta^v_i \delta^w_j \frac{[\mu + v]! [\nu - w]!}{[\nu - j]! [\mu + j]! [\mu - v]! [\mu - i]!} e^{-i \pi} (4ij - 2(v-j)(j-i) - (v-j)(v-j+1))$$

$$C^-: \quad \begin{array}{c} v \geq j, \\ i \geq w \end{array}$$
The Reshetikhin-Turaev formula for the colored Jones polynomial via Quantum R-matrices

Let $E, C, \cup_-, \cap_-$ and $K$ are the set of edges, crossings, cups, caps and kinks of a link diagram $D_L$ of a framed link $L$ and define

$$\mathcal{I} = \{j : E \mapsto \frac{1}{2}\mathbb{Z} \mid j(e) \in B_{\lambda_e} \forall e \in E, \ C_+, C_- \text{ satisfied } \forall c \in C\}.$$

The Jones polynomial of the framed link $L$ colored by $\lambda \in \Lambda_{\pi_0(L)}^\pi$ is given by

$$J^{(r)}_\lambda(L) = J_\lambda(L)(e^{2\pi i/r}) = \sum_{j \in \mathcal{I}} \left( \prod_{c \in C} C^{\epsilon(c)}(\lambda_{l(c)}, \lambda_{r(c)})^{j(e_{\text{in}}, -(c)); j(e_{\text{in}}, +(c))} \prod_{u \in \cup_-} e^{-\frac{2\pi i}{r}j(u)} \prod_{n \in \cap_-} e^{\frac{2\pi i}{r}j(n)} \prod_{k \in K} e^{\frac{\pi i}{r}e(k)(2\lambda(k)+1)(\lambda(k)+1)} \right).$$
The WRT quantum invariant via quantum R-matrices

For $\lambda \in \Lambda_0(L)$ and $\sigma(L) =$ signature of the framed link $L$, we set

$$[\lambda] = \left( \prod_{\ell \in \pi_0(L)} [\lambda_\ell] \right), \quad \star(L) = e^{\pi i \frac{3(r-2)}{r}} \sigma(L) \left( \sqrt{\frac{2}{r}} \sin \left( \frac{\pi}{r} \right) \right)^{|\pi_0(L)|+1}$$

**Theorem (Reshetikhin and Turaev)**

$$Z^{(r)}(M) = \star(L) \sum_{\lambda \in \Lambda_0(L)} [\lambda] J^{(r)}_{\lambda}(L).$$

is an invariant of $M = M_L$, the oriented closed three manifold obtained by doing surgery on the framed link $L$ in $S^3$.

Similarly, if $L = L_s \cup L_c$ and if we choose $\lambda \in \Lambda_0(L_c)$ then

$$Z^{(r)}(M_{L_s}, L_c, \lambda) = \star(L_s) \sum_{\mu \in \Lambda_0(L_s)} [\lambda] J^{(r)}_{(\mu, \lambda)}(L)$$

where $M_{L_s}$ is obtained by surgery on $L_s$. From now on we simply write

$$\star = \star(L)$$
Faddeev’s quantum dilogarithm

Recall Faddeev’s quantum dilogarithm $S_\gamma$ with parameter $\gamma$ given by

$$S_\gamma(z) = \exp\left(\frac{1}{4} \int_{\mathbb{R} + i\epsilon} \frac{e^{zy}}{\sinh(\pi y) \sinh(\gamma y) y} \, dy\right).$$

for $|\text{Re}(z)| < \gamma + \pi$, and $\gamma \in (0, 1)$.

- For $\text{Re}(\gamma) > 0$, $S_\gamma$ is a meromorphic function and $(1 + e^{iz})S_\gamma(z + \gamma) = S_\gamma(z - \gamma)$

$$S_{\pi b^2}(−2\pi ibz) = \frac{(wq, q^2)_{\infty}}{(\tilde{w}\tilde{q}, \tilde{q}^2)_{\infty}}, \quad w = e^{2\pi bx}, \quad q = e^{i\pi b^2}$$

$$\tilde{w} = e^{\frac{2\pi x}{b}}, \quad \tilde{q} = e^{-\frac{i\pi}{b^2}}, \quad \text{Im}(b^2) > 0$$

Introduce the following normalized version of the Faddeev’s quantum dilogarithm

$$\tilde{S}_\gamma(z) = \frac{S_\gamma(\pi - (2z + 1)\gamma)}{S_\gamma(\pi - \gamma)}.$$

We are going to set $\gamma = \pi / r$, then the function $\tilde{S}_\gamma$ has poles at $z = -1, -2, \ldots$ and zeros at $z = r, r + 1, \ldots$. In particular, we observe that

$$z \mapsto \frac{1}{\tilde{S}_\gamma(-z^2)}$$

is 1 in zero and zero on all other integers.

Further we have that

$$[m]! = \frac{e^{\frac{i\pi}{2r} m(m+1)}}{(2i \sin(\frac{\pi}{r}))^m} \tilde{S}_\gamma(m)$$
Meromorphic extension of the quantum R-matrix

**Definition**

\[ R^\pm \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4) \]

\[ R^+(y_1, y_2, z_1, z_2, z_3, z_4) = \frac{\tilde{S}_\gamma(y_1 + z_4)\tilde{S}_\gamma(y_2 - z_3)}{\tilde{S}_\gamma(z_4 - z_1)\tilde{S}_\gamma(y_1 + z_1)\tilde{S}_\gamma(y_2 - z_2)} \times e^{i\pi Q_+/(y_1, y_2, z_1, z_2, z_3, z_4)} \]

\[ \tilde{S}_\gamma(-(z_1 + z_2 - z_3 - z_4)^2) \]

\[ R^-(y_1, y_2, z_1, z_2, z_3, z_4) = \frac{\tilde{S}_\gamma(y_2 + z_3)\tilde{S}_\gamma(y_1 - z_4)}{\tilde{S}_\gamma(z_3 - z_2)\tilde{S}_\gamma(y_2 + z_2)\tilde{S}_\gamma(y_1 - z_1)} \times e^{i\pi Q_-/(y_1, y_2, z_1, z_2, z_3, z_4)} \]

\[ \tilde{S}_\gamma(-(z_1 + z_2 - z_3 - z_4)^2) \]

Here \( Q_\pm \) are explicit quadratic fn’s of \( y_1, y_2, z_1, z_2, z_3, z_4 \).

**Lemma**

\[ C^\pm(\nu, \mu)_{i;j}^{v;w} = R^\pm(\nu, \mu, i, j, v, w) \]
Definition

Let $J(D_L)_r \in \mathcal{M}(\mathbb{C}^C \times \mathbb{C}^E)$ be given by

$$J(D_L)_r(y, z) = \prod_{c \in C} R^e(c)(y_l(c), y_r(c), z_{e_{\text{in}}, -(c)}, z_{e_{\text{in}}, +(c)}, z_{e_{\text{out}}, -(c)}, z_{e_{\text{out}}, +(c)}) \prod_{e \in E} \cot(\pi(z_e + y_e))$$

Let $\Gamma_{\lambda}$ be a contour in the complex plane close to the real axis, which encircles $\{-\lambda, -\lambda + 1, \ldots, \lambda - 1, \lambda\}$.

- $\Gamma_{D_L} = \prod_{e \in E} \Gamma_{\lambda(e)} \subset \mathbb{C}^E$.

Theorem (A. & Mistegaard)

$$J^{(r)}_{\lambda}(L) = \int_{z \in \Gamma_{D_L}} J(D_L)_r(\lambda, z) \bigwedge_{e \in E} dz_e$$

- $\widehat{\Gamma}_r \subset \mathbb{C}$ is a contour in $\mathbb{C}$ close to the real axis, which encircles $\{1, \ldots, r\}$.

Theorem (A. & Mistegaard)

$$Z^{(r)}(M_L) = \ast \int_{y \in \widehat{\Gamma}_r^{\pi_0(L)}} J^{(r)}_y(D_L) \prod_{\gamma \in \pi_0(L)} \frac{\sin\left(\frac{(2y_\gamma + 1)\pi}{r}\right)}{\sin\left(\frac{\pi}{r}\right)} \cot(2\pi y_\gamma) \bigwedge_{\gamma \in \pi_0(L)} dy_\gamma$$
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in this path integral setting.
Finite dimensional model

We have just derived the following finite dimensional model

\[ Z^{(r)}(M_L) = \star \int_{(y,z) \in \tilde{\Gamma}_{r0}(L) \times \Gamma_{DL}} Z(D_L)_r(y,z) \wedge_{\gamma \in \pi_0(L)} dy_{\gamma} \wedge_{e \in E} dz_e \]

where

\[ Z(D_L)_r(y,z) = J(D_L)_r(y,z) \prod_{\gamma \in \pi_0(L)} \frac{\sin\left(\frac{(2y_{\gamma}+1)\pi}{r}\right)}{\sin\left(\frac{\pi}{r}\right)} \cot(2\pi y_{\gamma}) \]

\[ \tilde{\Gamma}_{r0}(L) \times \Gamma_{DL} \subset \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E \]

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

\[ \star = e^{\pi i \frac{3(r-2)}{r} \sigma(L)} \left( \sqrt{\frac{2}{r}} \sin\left(\frac{\pi}{r}\right) \right)^{|\pi_0(L)|+1} \]
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\]

**Question:** In what sense might

\[
J(D_L)^r, Z(D_L)^r \in \mathcal{M}(\mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E)
\]

be an invariant of \(L\) and \(M_L\) respectively?
Conjectures about $Z^{(r)}$ and resurgence as a Rosetta stone

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.

$$Z^{(r)}(M) \sim_{r \to \infty} \sum_{\theta \in \text{CS}_{\text{SU}(2)}} \exp(2\pi i r \theta) r^{d \theta} b_\theta (1 + Z_\theta (r^{-1})).$$

where $Z_\theta (r^{-1}) = c_\theta^{(1)} r^{-1} + c_\theta^{(2)} r^{-2} + \ldots$.

Here $\sim_{r \to \infty}$ means

$$\left| Z^{(r)}(M) - \sum_{\theta \in \text{CS}_{\text{SU}(2)}} \exp(2\pi i r \theta) r^{d \theta} b_\theta (1 + Z_\theta^L (r^{-1})) \right| \leq C_L r^{d-L-1} \quad \forall r \in \{2, 3 \ldots\},$$

where $C_L$ are some constants and

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2. The volume conjecture: relating $Z^{(r)}$ to hyperbolic geometry.

*Kashaev’s original volume conjecture (in the MM formulation):*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \left( \frac{|J_\lambda(K)(e^{2\pi i/\lambda})|}{|J_\lambda(U)(e^{2\pi i/\lambda})|} \right) = \frac{1}{2\pi} \text{Vol}(S^3 - K)$$
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Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.
2. The volume conjecture: relating $Z^{(r)}$ to hyperbolic geometry.
3. Integrality and categorification of $Z^{(r)}$: The GPPV invariant of a 3-mfd. $M$ (defined as a string theory BPS index) with a spin$^c$ structure $a$ is an essential integer power series

$$\hat{Z}_a(M; q) \in 2^{-c} q^{\Delta_a} \mathbb{Z}[[q]],$$

where $\Delta_a \in \mathbb{Q}$, $c \in \mathbb{Z}_+$. Conjecture: There exists a homology theory $H_{\bullet, \bullet}(M; a)$ s.t.

$$\hat{Z}_a(M; q) = 2^{-c} q^{\Delta_a} \sum_{i,j} (-1)^i q^j \dim (H_{i,j}(M; a)).$$
Conjectures about $Z^{(r)}$ and resurgence as a Rosetta stone

**Main conjectures concerning the WRT-TQFT:**

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2. **The volume conjecture:** relating $Z^{(r)}$ to hyperbolic geometry.
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\hat{Z}_a(M; q) = 2^{-c} q^{\Delta_a} \sum_{i,j} (-1)^i q^j \dim (H_{i,j}(M; a)).
$$

The asymptotic expansion of the TQFT $Z^{(r)}$, the volume conjecture and $\hat{Z}_a(M; q)$ are connected via resurgence.
Main conjectures concerning the WRT-TQFT:

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Divergent series: In mathematical physics divergent series are common and many examples come from path integrals.

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Resummation of divergent power series

- **Divergent series**: In mathematical physics divergent series are common and many examples come from path integrals.

- **Problem**: Given a divergent power series

\[ \varphi(z) \in z^{-1} \mathbb{C}[[z^{-1}]]. \]

we want to construct a holomorphic function ($D = \text{domain in } \mathbb{C}$ such that \( \infty \in \overline{D} \))

\[ \hat{\varphi} \in \mathcal{O}(D) \]

having \( \varphi \) as an asymptotic expansion, i.e. \( \forall m \in \mathbb{N} \)

\[ \hat{\varphi}(z) = \varphi_0 z^{-1} + \cdots + \varphi_m z^{-m-1} + O(z^{-m-2}). \]

- **Borel-Laplace resummation** is (sometimes) a solution to this problem.
The Borel transform $\mathcal{B}$

Definition

The Borel transform

$$\mathcal{B} : z^{-1} \mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$$

is the $\mathbb{C}$-linear extension of

$$\mathcal{B} (z^{-\alpha - 1}) = \frac{\zeta^\alpha}{\Gamma(\alpha + 1)} = \frac{\zeta^\alpha}{\alpha!}.$$

Here $\Gamma$ is the Gamma function, which for $\text{Re}(x) > 0$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.$$
The Borel transform $\mathcal{B}$ as the inverse of the Laplace transform $\mathcal{L}$

- **The Laplace transform** $\mathcal{L}$: Let $\gamma \subset \mathbb{C}$ be an oriented countour. Let $g$ be a holomorphic function. Define

$$\mathcal{L}_\gamma(g)(z) = \int_\gamma \exp(-z \cdot \zeta)g(\zeta) \, d\zeta.$$  

when ever this integral converges say absolutely.

**Proposition**

For all $m \in \mathbb{N}$ one has

$$\mathcal{L}_{\mathbb{R}^+} \circ \mathcal{B}(z^{-m-1}) = z^{-m-1},$$

$$\mathcal{B} \circ \mathcal{L}_{\mathbb{R}^+}(\zeta^m) = \zeta^m.$$
Proposition

Let \( \varphi(z) \in z^{-1} \mathbb{C}[[z^{-1}]] \). Assume \( B(\varphi)(\zeta) \) extends to an analytic function of appropriate bound along \( \gamma(\theta) = \exp(i\theta) \mathbb{R}_+ \). Consider

\[
\tilde{\varphi}(z) \overset{\text{def.}}{=} \mathcal{L}_{\gamma(\theta)} \circ B(\varphi)(z) = \int_{\gamma(\theta)} e^{-\zeta z} B(\varphi)(\zeta) \, d\zeta.
\]
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\]

The function \( \hat{\varphi}(z) \) is analytic on \( \text{Re}(z \exp(i\theta)) > 0 \) and has \( \varphi(z) \) as Poincaré asymptotic expansion

\[
\hat{\varphi}(z) \sim_{|z| \to \infty} \varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]].
\]
Picard-Lefshetz theory and resurgence

- **Picard-Lefshetz theory and resurgence**: Let $f \in \mathcal{O}(X^d)$, let $\omega \in \Omega^d_{\text{Hol}}(X^d)$. Let $\lambda \in \mathbb{C}^*$. Let $\Delta_\varphi$, $\varphi = \text{arg}(\lambda)$ be a PL-thimble emanating from a critical point $z_c$ of $f$ and consider

$$I_{\Delta_\varphi}(\lambda) = \int_{\Delta_\varphi} \exp(-\lambda f(z)) \omega(z)$$

Think of the $t$-plane as the set of values of $f$ with a discrete set of critical values (here $0, t_+, t_-$) with curves $\gamma$ emanating from them, along which the exponential factor is decaying.
Illustration of a Picard-Lefschetz thimble $\Delta(\sigma, \gamma)$

Below we illustrate a Picard-Lefshetz thimble $\Delta_{\varphi}(\sigma)$ foliated by vanishing cycles $\sigma(t) \in H_*(f^{-1}(t), \mathbb{Z})$ which are parallel (w.r.t. the Gauss-Manin connection) along a curve $\gamma \subset \text{Im}(f)$.

Figure: Thimble $\Delta_{\varphi}(\sigma)$ in $d = 2$. 
Picard-Lefshetz theory and resurgence: Let $f \in \mathcal{O}(X^d)$, let $\omega \in \Omega^d_{\text{Hol}}(X^d)$, let $\Delta \varphi$ be a PL-thimble emanating from a critical point $z_c$ and consider

$$I_{\Delta \varphi}(\lambda) = \int_{\Delta \varphi} \exp(-\lambda f(z)) \omega(z).$$

Let $\tilde{I}_{\Delta \varphi} \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$ be such that we have the asymptotic expansion

$$I_{\Delta \varphi}(\lambda) \sim \exp(-\lambda f(z_{\Delta \varphi})) \lambda^d \Delta \varphi (1 + \tilde{I}_{\Delta \varphi}(\lambda^{-1})).$$

The aim of Écalle’s theory of resurgence is to decode the information contained in the divergent series

$$\tilde{I}_{\Delta \varphi} \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$$

and determine the properties of the Borel resummation of them as analytic multi-value functions and in turn to recover from them the actual integrals $I_{\Delta \varphi}(\lambda)$. 

Jørgen Ellegaard Andersen  
Quantum Chern-Simons Theory and Resurgence
Picard-Lefshetz theory and resurgence

In particular, when we turn the argument $\varphi$ of the expansion parameter $\lambda$ around, then the $\gamma$’s emanating from some critical $z_c$ point will hit some other critical point $z'_c$, say at $\varphi_0$ and at this point the corresponding $I_{\Delta \varphi}$ jumps to

$$I_{\Delta \varphi_0 + \epsilon} = I_{\Delta \varphi_0 - \epsilon} + n(z_c, z'_c) I_{\Delta' \varphi_0},$$

hence we see that

$I_{\Delta' \varphi_0}$ resurges with a multiplicity factor $n(z_c, z'_c)$ in $I_{\Delta \varphi_0}$.

Écalle has developed his Alien calculus precisely to determine these jumping or wall crossing phenomenon directly from the divergent series $\tilde{I}_{\Delta \varphi}$.

There are new reformulations of parts of Écalle’s Alien calculus, such as

- **Analytic Wall Crossing Structure** by Kontsevich-Soibelman
- Ongoing work of A. and Kontsevich on **Cycle Systems and Analytic Realizations**.

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*Jørgen Ellegaard Andersen*  
*Quantum Chern-Simons Theory and Resurgence*
Resurgence in TQFT: In a number of different publications Witten, Garoufalidis, Gukov-Marino-Putrov and Gukov-Pei-Putrov-Vafa, Garoufalidis-Gu-Marino-Wheller have proposed/argued one can in lots of examples apply the theory of Resurgence to the partition function of SU(2) Chern-Simons theory \((r = k + 2)\)

\[
Z^{(r)}(M) = \int_{A \in A_{\text{SU}(2)}/G_{\text{SU}(2)}} \exp\left(2\pi ikC(A)\right) D A
\]

by thinking of

\[
A_{\text{SU}(2)}/G_{\text{SU}(2)} \subset A_{\text{SL}(2,\mathbb{C})}/G_{\text{SL}(2,\mathbb{C})}.
\]

as a real "middle dimensional cycle" and decomposing it into

"middle dimensional Picard-Lefshetz thimbles"

in this path integral setting.
Finite dimensional model

We have just derived the following finite dimensional model

\[ Z^{(r)}(ML) = \star \int_{(y,z) \in \tilde{\Gamma}_r \times \Gamma_{DL}} Z(D_L)(y,z) \bigwedge_{\gamma \in \pi_0(L)} dy_\gamma \bigwedge_{e \in E} dz_e \]

where

\[ Z(D_L)(y,z) = J(D_L)(y,z) \prod_{\gamma \in \pi_0(L)} \frac{\sin\left(\frac{(2y_\gamma+1)\pi}{r}\right)}{\sin\left(\frac{\pi}{r}\right)} \cot(2\pi y_\gamma) \]

\[ \tilde{\Gamma}_r \times \Gamma_{DL} \subset \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E \]

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

\[ \star = e^{\pi i \frac{3(r-2)}{r} \sigma(L)} \left( \sqrt{\frac{2}{r}} \sin\left(\frac{\pi}{r}\right) \right)^{|\pi_0(L)|+1} \]
Conjectures about $Z^{(r)}$ and resurgence as a Rosetta stone

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating $Z^{(r)}$ to classical Chern-Simons theory.

$$Z^{(r)}(M) \sim_{r \to \infty} \sum_{\theta \in \text{CS}_{\text{SU}(2)}} \exp(2\pi ir\theta) r^{d_\theta} b_\theta (1 + Z_\theta (r^{-1})).$$

where $Z_\theta (r^{-1}) = c_\theta^{(1)} r^{-1} + c_\theta^{(2)} r^{-2} + \ldots$.

Here $\sim_{r \to \infty}$ means

$$\left| Z^{(r)}(M) - \sum_{\theta \in \text{CS}_{\text{SU}(2)}} \exp(2\pi ir\theta) r^{d_\theta} b_\theta (1 + Z_\theta^L (r^{-1})) \right| \leq C_L r^{d - L - 1} \quad \forall r \in \{2, 3 \ldots \},$$

where $C_L$ are some constants and

$$d = \max \{d_\theta\}, \quad Z_\theta^L (r^{-1}) = \sum_{\ell=1}^{L} c_\theta^{(\ell)} r^{-\ell}$$
The Resurgence conjecture as a Rosetta stone

**The Resurgence Conjecture:**
(Garoufalidis; Gukov, Marino, Putrov; Gukov, Pei, Putrov, Vafa; A. & Mistegaard)

1. **Borel Resummability:** The series $Z_\theta$ are Borel resummable, $\mathcal{B}(Z_\theta)$ endless continuable.

2. **Generalised Volume Conjecture:** The set of poles $\Omega(\theta)$ of the meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies that

   $$-2\pi i \text{CS}_C(M) = \bigcup_\theta (\Omega(\theta) - 2\pi i \theta) \mod Z.$$

3. **Wall Crossing:** The meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies and are in part determined by a Wall crossing structure.

4. **The $\hat{Z}$-Conjecture:** The $\hat{Z}_\alpha$ GPPV invariants can be obtained from the functions $\mathcal{B}(Z_\theta)$ by a finite Laplace type transform.

5. **AK-TQFT:** $\theta_c = \text{CS}$ of conjugate representation: $\mathcal{L} \circ \mathcal{B}(Z_{\theta_c}) = Z^{AK}$.

6. **The Radial Limit Conjecture:** The WRT invariant $Z^{(r)}$ can be (re)-obtained from $\hat{Z}_0$ as a limit $Z^{(r)} = \frac{1}{\sqrt{r}} \lim_{q \to e^{2\pi i/r}} \hat{Z}_0(q)$.

Here $Z^{AK} =$ Andersen-Kashaev TQFT.
Topological invariance and the definition of $\hat{Z}_a(Y; q)$

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**Definition (Gukov, Pei, Putrov, Vafa)**

Let $\Gamma$ be a plumbing graph. Then there is an explicit definition of

$$\Delta_a \in \mathbb{Q}, \; c \in \mathbb{Z}_+, \; \hat{Z}_a(\Gamma; q) \in 2^{-c} q^{\Delta_a} \mathbb{Z}[[q]],$$

for each spin$_c$-structure $a$ on $Y_{\Gamma}$.

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**Theorem (Gukov, Manolescu)**

If $Y_{\Gamma} = Y'_{\Gamma}$ (e.g. $\Gamma$ and $\Gamma'$ are related by Neumann moves) then

$$\hat{Z}_a(\Gamma; q) = \hat{Z}_a(\Gamma'; q).$$
The AK-TQFT

Theorem (Andersen & Kashaev)

For any hyperbolic knot $K$ the AK-TQFT is determined by a Jones function

$$J_{K,h} \in S(\mathbb{R})$$

parametrised by $h \in \mathbb{R}^+$, which is topological invariant of the knot.

Conjecture (AK-Volume conjecture)

The hyperbolic volume of $S^3 - K$ is recovered as the following limit

$$\lim_{h \to 0} 2\pi h \log |J_{K,h}(0)| = -\text{vol}(S^3 - K).$$

Examples: ($\Phi_b(z) = S\pi b^2 (-2\pi i bz)$ and $h = (b + b^{-1})^{-2}$)

$$J_{41,h}(x) = \int_{\mathbb{R} - i\epsilon} \frac{\Phi_b(x - y)}{\Phi_b(y)} e^{2\pi i x(2y - x)} \, dy$$

$$J_{52,h}(x) = e^{-i\pi/3} \int_{\mathbb{R} - i\epsilon} \frac{e^{i\pi(z - x)(z + x)}}{\Phi_b(z + x)\Phi_b(z - x)\Phi_b(z)} \, dz.$$
The AK-TQFT

Let \( z = e^{2\pi bx}, \quad q = e^{2\pi ib^2}, \quad \tilde{q} = e^{-2\pi ib^{-2}} \) and \( \tilde{z} = e^{2\pi b^{-1}x} \).

Introduce

\[
g(z, q) = h(z, q)^{-1} H^+(z, q)
\]

\[
G(z, q) = h(z, q) H^-(z, q)
\]

where

\[
h(z, q) = \frac{(q, q)_\infty (-z q^{1/2}, q)_\infty^2 (-z^{-1} q^{1/2}, q)_\infty^2}{(zq, q)_\infty}
\]

\[
H^+(z, q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2} z^{2m+1}}{(q, q)_m (zq, q)_m}
\]

\[
H^-(z, q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2} z^{-m+1}}{(q, q)_m (z^{-1} q, q)_m (1-z)}
\]

\[
J_{4,\tilde{h}}(x) = \frac{q^{\frac{1}{12}} \tilde{q}^{-\frac{1}{12}}}{2\pi b} g(z, q)G(\tilde{z}, \tilde{q}) - \frac{q^{-\frac{1}{12}} \tilde{q}^{\frac{1}{12}}}{2\pi b^{-1}} G(z, q)g(\tilde{z}, \tilde{q})
\]
Results: (Joint with William Elbaek Mistegaard)

Resurgence Analysis of WRT-Invariants of SF-manifolds

For a Seifert homology sphere $M = \Sigma(p_1, \ldots, p_n)$ we prove:

1. A decomposition and identification

$$\pi_0(\mathcal{M}(\text{SL}(2, \mathbb{R})) \cup \mathcal{M}(\text{SU}(2))) \cong \pi_0(\mathcal{M}(\text{SL}(2, \mathbb{C}))) \cong \text{CS}_\mathbb{C}(X).$$

2. $\hat{Z}_0(q)$ is a resummation of the Ohtsuki series $Z_0$

$$\hat{Z}_0(q) = \frac{1}{\sqrt{\tau}} \mathcal{L} \circ \mathcal{B}(Z_0)(1/\tau), \quad (q = \exp 2\pi i \tau).$$

3. A full asymptotic expansion of $\hat{Z}_0(q)$ for $\tau$ near $1/(r - 2)$ implying

$$Z^{(r)}(M) = \frac{1}{\sqrt{k}} \lim_{q \to e^{2\pi i/(r-2)}} \hat{Z}_0(q).$$

4. An identification of the poles $\Omega$ of the Borel transform $\mathcal{B}(Z_0)$

$$-2\pi i \text{CS}_\mathbb{C}^*(X) + 2\pi i \mathbb{Z} = \Omega.$$

Our work builds on work of Lawrence and Rozansky on $Z^{(r)}(X)$ and is inspired by work of Gukov, Marino and Putrov.
Hyperbolic surgeries on the figure 8 knot

We now present in more detail the analysis leading to the above results for the hyperbolic 3-manifolds

\[ M(4_1(a/b)) = M_{a/b} \]

with surgery link giving by the figure eight knot with surgery data \( a/b \).

Figure: Figure eight knot 4_1
Choose $c, d \in \mathbb{Z}$ with $ad - cb = 1$. Define

$$\chi_{n,r}(x, y) = \sin\left(\frac{\pi}{b}(x - nd)\right) e^{2\pi ir\left(\frac{dn^2}{b} + \frac{a}{4b}x^2 - \frac{n}{b}x - xy\right)}$$

$$\times \frac{S_\gamma(-\pi + 2\pi(x-y))}{S_\gamma(-\pi + 2\pi(x+y))} \cot(\pi rx) \tan(\pi ry).$$

and let

$$\Omega^{a,b}_r(a, y) = \sum_{n \in \mathbb{Z}/|b|\mathbb{Z}} \chi_{n,r}(x, y)$$

Joint with Hansen we proved

$$Z^{(r)}(M_{a/b}) = c_1 r e^{\frac{2\pi i}{r} c_2} \int_{C_1(r) \times C_2(r)} \Omega^{a,b}_r(a, y) \, dy \, dx$$

where $c_1, c_2 \in \mathbb{C}^*$ and $C_1(r)$ is a simple closed contour which encircles the set \{m/r : m = 1, 2, ..., r - 1\}, and $C_2(r)$ is a simple closed contour encircling \{(m + 1/2)/r : m = 0, 1, ..., r - 1\}. 
The semiclassical asymptotics of $S_\gamma$ is given by Euler’s dilogarithm: For $\text{Re}(z) < \pi$ we have

$$S_\gamma(z) = \exp\left(\frac{r}{2\pi i} \text{Li}_2 (-e^{i z}) + I_\gamma(z)\right),$$

$$|I_\gamma(z)| = O(\gamma)$$

This leads us to the following phase functions indexed by $\alpha, \beta \in \{0, 1\}$ and $n \in \mathbb{Z}/|b|\mathbb{Z}$

$$\Phi_{n}^{\alpha,\beta}(x, y) = \frac{\text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i(x-y)})}{4\pi^2} - \frac{dn^2}{b} + \left(-\frac{a}{4b}x + \frac{n}{b} + y + \alpha + \beta\right)x + y(\alpha - \beta).$$
Identification of classical complex Chern-Simons values

Theorem (A. & Hansen)

There exists a surjection

$$(x, y) \mapsto [\rho_{x,y}]$$

from the set of critical points $(x, y)$ of the phase functions $\Phi_{n}^{\alpha, \beta}$ with $x \notin \mathbb{Z}$ onto $\mathcal{M}^{*}(M_{a/b}, \text{SL}(2, \mathbb{C}))$. Moreover, we have that

$$\Phi_{n}^{\alpha, \beta}(x, y) = \text{SCS}([\rho_{x,y}]) \mod \mathbb{Z}.$$

In the variables

$$v = e^{\pi i x}, \quad w = e^{2 \pi i y}$$

the critical point equations are

$$v^{-a} = \left( \frac{w - v^2}{1 - v^2 w} \right)^b, \quad (1 - v^2 w)(w - v^2) = v^2 w.$$

There are finitely many solutions $\{(v_{\theta}, w_{\theta})\}$, $\theta \in \text{CS}_{\mathbb{C}}(M_{a/b})$. 

Jørgen Ellegaard Andersen
Quantum Chern-Simons Theory and Resurgence
A resurgence Theorem

Based on generalizations of resurgence results for Laplace integrals due to Malgrange and Pham we prove

**Theorem (A. & Mistegaard)**

There exists power series

\[ \{Z_\theta(x)\} \theta \in \text{CS} \subset \mathbb{C}[[x^{-1}]] \]

giving a full asymptotic expansion

\[ Z^{(r)}(M_{a/b}) \sim_{r \to \infty} r \sum_{\theta \in \text{CS}} e^{2\pi ir\theta} Z_\theta(r). \]

Each Borel transform \( B(Z_\theta) \) is resurgent with singularities \( \Omega(\theta) \) s.t.

\[ \bigcup_\theta (\Omega(\theta) - 2\pi i \theta) = -2\pi i \text{CS}_\mathbb{C} + 2\pi i \mathbb{Z}. \]

Let \( \Gamma_{\theta}^\hbar \) be the Lefschetz thimble at \((v_\theta, w_\theta)\), then

\[ \mathcal{L} \circ B(Z_\theta)(\hbar) = 2\pi ic_1 \frac{e^{\hbar c_2}}{\hbar} \int_{\Gamma_{\theta}^\hbar} \frac{\Omega_{a,b}^{\frac{2\pi i}{\hbar}}(x, y)}{\hbar} \, dy \, dx \]
Yoon’s generalization of the Cho-Murakami potential

\[ W_+ (m_l, m_r, w_l, w_r, w_u, w_d) \]
\[ = \text{Li}_2(e^{2\pi i(w_l - w_d - m_r)}) + \text{Li}_2(e^{2\pi i(w_r - w_d - m_l)}) \]
\[ - \text{Li}_2(e^{2\pi i(w_u - w_r - m_r)}) - \text{Li}_2(e^{2\pi i(w_u - w_l - m_l)}) - \frac{\pi^2}{6} \]
\[ + \text{Li}_2(e^{2\pi i(w_d - w_l + w_u - w_r)}) + 4\pi^2(w_l - w_d - m_r)(w_r - w_d - m_l), \]

and

\[ W_- (m_l, m_r, w_l, w_r, w_u, w_d) \]
\[ = \text{Li}_2(e^{2\pi i(w_u - w_r + m_r)}) + \text{Li}_2(e^{2\pi i(w_u - w_l + m_l)}) \]
\[ - \text{Li}_2(e^{2\pi i(w_l - w_d + m_r)}) - \text{Li}_2(e^{2\pi i(w_r - w_d + m_l)}) + \frac{\pi^2}{6} \]
\[ - \text{Li}_2(e^{2\pi i(w_d - w_l + w_u - w_r)}) + 4\pi^2(w_l - w_d + m_r)(w_r - w_d + m_l). \]
Let $D_L$ be a link diagram, $C$ the set of crossings and $F$ the set of complementary regions of $L$.

For each crossing $c$ consider

$$
\pi_c : \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \rightarrow \mathbb{C}^2 \times \mathbb{C}^4.
$$

We define the pre-potential function $W \in \tilde{O}(\mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F)$ by

$$
W = \sum_{c \in C} \mathcal{W}_{\epsilon(c)} \circ \pi_c.
$$

The Yoon-Cho-Murakami potential function of the link $L$ is

$$
\tilde{W}(y, w) = W(y, w) - \sum_{f \in F} \frac{\partial W}{\partial w_f}(y, w)w_f - \sum_{\ell \in \pi_0(L)} \frac{\partial W}{\partial y_\ell}(y, w)y_\ell.
$$
Define

\[ S' = \{(y, w) \in \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \mid \frac{\partial W}{\partial w_f} \in 4\pi^2\mathbb{Z}, \forall f \in F; \]
\[
\frac{\partial W}{\partial y_\ell} \in 4\pi^2\mathbb{Z}, y_\ell \notin \mathbb{Z}, \forall \ell \in \pi_0(L)\}\].

and

\[ M' = \{\rho \in \text{hom}(\pi_1(M), \text{PSL}(2, \mathbb{C})) \mid \rho(\mu_\ell) \neq \pm \text{Id}, \ell \in \pi_0(L)\}/\text{PSL}(2, \mathbb{C}).\]

**Theorem (Yoon)**

*There is a surjective holomorphic map* \( \rho : S' \to M' \), *and we have*

\[ -i S_{\text{CS}\Sigma} \circ \rho = \tilde{W} \mod \pi^2\mathbb{Z}. \]
Finite dimensional model

We have just derived the following finite dimensional model

\[ Z^{(r)}(M_L) = \star \int_{(y,z) \in \tilde{\Gamma}_r \times \Gamma_{DL}} \left( \prod_{\gamma \in \pi_0(L)} \frac{\sin(\frac{(2y+1)\pi}{r})}{\sin(\frac{\pi}{r})} \cot(2\pi y) \right) dy_{\gamma} \wedge dz_e \]

where

\[ Z(D_L)(y,z) = J(D_L)(y,z) \prod_{\gamma \in \pi_0(L)} \frac{\sin(\frac{(2y+1)\pi}{r})}{\sin(\frac{\pi}{r})} \cot(2\pi y) \]

\[ \tilde{\Gamma}_r \times \Gamma_{DL} \subset \mathbb{C} \pi_0(L) \times \mathbb{C} E \]

is a middle dimensional cycle in this finite dimensional affine space.

Recall that

\[ \star = e^{\pi i \frac{3(r-2)}{r}} \sigma(L) \left( \sqrt{\frac{2}{r}} \sin(\frac{\pi}{r}) \right)^{|\pi_0(L)|+1} \]
If we now apply the semiclassical asymptotics of $S_\gamma$:

For $\text{Re}(z) < \pi$ we have ($\gamma = \frac{\pi}{r}$)

$$S_\gamma(z) = \exp\left(\frac{r}{2\pi i} \text{Li}_2\left(-e^{iz}\right) + O(\gamma)\right)$$

**Theorem (A. & Mistegaard)**

For $D_L$ there is a map

$$\Psi : \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^F \to \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^E$$

such that when one applies the above semiclassical approximation to all $S_\gamma$ in $J(D_L)$ one obtains the Yoon-Cho-Murakami potential function $\tilde{W}$ as the phase function, e.g.

$$J(D_L)(\Psi(y, w)) = e^{\frac{r}{2\pi i} \tilde{W}(y, w) + O(\gamma)}$$

for all $(y, w) \in \Psi^{-1}(\tilde{\Gamma}_r^{\pi_0(L)} \times \Gamma_{D_L})$.

The map $\Psi$ is the identity on the first factor and on the second factor it is given by

$$ze = w_{f_-}(e) - w_{f_+}(e), \quad \forall e \in E$$
Conjecture

There exist finitely many
- "abelian" contours $\Gamma_a$, and coefficients $n_a \in \mathbb{Z}$, $a \in \text{CS}_{ab}(M)$ and
- Lefschetz thimbles $\Gamma_\theta$ and coefficients $n_\theta \in \mathbb{Z}$, $\theta \in \text{CS}_{non-ab}(M)$ such that

\[
Z^{(r)}(M) = \sum_{a \in \text{CS}_{ab}(M)} n_a \int_{\Gamma_a} Z(D_L)(y, z) \wedge \left( \sum_{\gamma \in \pi_0(L)} dy_{\gamma} \wedge \left( \sum_{e \in E} dz_e \right) \right)
+ \sum_{\theta \in \text{CS}_{non-ab}(M)} n_\theta \int_{\Gamma_\theta} Z(D_L)(y, z) \wedge \left( \sum_{\gamma \in \pi_0(L)} dy_{\gamma} \wedge \left( \sum_{e \in E} dz_e \right) \right)
\]

(Work in progress on this conjecture joint with Maxim Kontsevich)
Theorem (A. & Kontsevich)

**If Conjecture 1 is true for some link \( L \) then:**

1. There exists power series

\[
\{Z_\theta(x)\}_{\theta \in CS} \subset \mathbb{C}[[x^{-1}]]
\]

giving a full asymptotic expansion

\[
\tau_r(M_L) \sim_{r \to \infty} r^d \sum_{\theta \in CS} e^{2\pi i r \theta} Z_\theta(r).
\]

2. Each Borel transform \( B(Z_\theta) \) is resurgent with singularities \( \Omega(\theta) \) s.t.

\[
\bigcup_{\theta} (\Omega(\theta) - 2\pi i \theta) = -2\pi i \text{CS}_\mathbb{C}(M_L) + 2\pi i \mathbb{Z}.
\]

3. We have that

\[
\mathcal{L} \circ B(Z_\theta) = \int_{\Gamma_\theta} Z(D_L)(y, z) \bigwedge_{e \in E} dz_e \bigwedge_{\ell \in \pi_0(L)} dy_\ell
\]
The Resurgence conjecture as a Rosetta stone

The Resurgence Conjecture:
(Garoufalidis; Gukov, Marino, Putrov; Gukov, Pei, Putrov, Vafa; A. & Mistegaard)

1. Borel Resummability: The series $Z_\theta$ are Borel resummable, $B(Z_\theta)$ endless
continuable.

2. Generalised Volume Conjecture: The set of poles $\Omega(\theta)$ of the meromorphic
functions $B(Z_\theta)$ satisfies that

$$-2\pi i \, \text{CS}_C(M) = \bigcup_\theta (\Omega(\theta) - 2\pi i \theta) \mod \mathbb{Z}.$$ 

3. Wall Crossing: The meromorphic functions $B(Z_\theta)$ satisfies and are in part
determined by a Wall crossing structure.

4. The $\hat{Z}$-Conjecture: The $\hat{Z}_a$ GPPV invariants can be obtained from the functions
$B(Z_\theta)$ by a finite Laplace type transform.

5. AK-TQFT: $\theta_c = \text{CS}$ of conjugate representation: $\mathcal{L} \circ B(Z_{\theta_c}) = Z^{AK}$.

6. The Radial Limit Conjecture: The WRT invariant $Z^{(r)}$ can be (re)-obtained from
$\hat{Z}_0$ as a limit $Z^{(r)} = \frac{1}{\sqrt{r}} \lim_{q \to e^{2\pi i/r}} \hat{Z}_0(q)$.

Here $Z^{AK} =$ Andersen-Kashaev TQFT.
A Seifert fibered homology 3-sphere $X = \Sigma(p_1, \ldots, p_n)$

Let $p_1, \ldots, p_n \in \mathbb{N}$, $n \geq 3$ be pairwise coprime and consider the Seifert fibered homology 3-sphere with $n \geq 3$ exceptional fibers

$$X = \Sigma(p_1, \ldots, p_n).$$

Figure: Surgery link for $X$. 
Complex Chern-Simons theory on $X$

For $x \in \mathbb{Q}$ let $[x] = x \mod \mathbb{Z}$. Set $P = p_1 \cdots p_n$. We prove

**Theorem (A. & Mistegaard)**

*The Chern-Simons action is injective on $\pi_0(\mathcal{M}(\text{SL}(2, \mathbb{C})))$ and*

$$\text{CS}^*_C(X) = \left\{ \begin{bmatrix} -m^2 \\ 4P \end{bmatrix} : m \in \mathbb{Z} \text{ is divisible by at most } n - 3 \text{ of the } p_j \text{'s} \right\}.$$  

*The natural inclusion $\mathcal{M}(\text{SL}(2, \mathbb{R})) \cup \mathcal{M}(\text{SU}(2)) \to \mathcal{M}(\text{SL}(2, \mathbb{C}))$ induces an isomorphism on the level of $\pi_0$*

$$\pi_0(\mathcal{M}(\text{SL}(2, \mathbb{R})) \cup_{\mathcal{M}(U(1))} \mathcal{M}(\text{SU}(2))) \cong \pi_0(\mathcal{M}(\text{SL}(2, \mathbb{C}))).$$
Introduce the rational function

\[
G(z) = \prod_{j=1}^{n} \left( \frac{P}{z^P - z^{-P}} \right) = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]].
\]
Introduce the rational function

\[ G(z) = \prod_{j=1}^{n} \left( \frac{P}{z^{p_j}} - z \frac{P}{p_j} \right) = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]]. \]

**Theorem (A. & Mistegaard)**

1. The Borel transform \( B(Z_0) \) is the function

\[ B(Z_0)(\zeta) = \frac{4\sqrt{2}\pi i P}{\pi i \sqrt{\zeta}} G \left( \exp \left( \frac{c\sqrt{\zeta}}{P} \right) \right). \]

2. Let \( \Omega \) be the set of poles of \( B(Z_0) \). Then

\[ \frac{i}{2\pi} \Omega = CS_\mathbb{C}^*(X) + \mathbb{Z}. \]
The resurgence formula $\hat{Z}_0 = \mathcal{L} \circ \mathcal{B}(Z_0)$

**Theorem (A. & Mistegaard)**

Set $q = \exp(2\pi i \tau), \tau \in \mathbb{H}$. We have

$$\hat{Z}_0(q) = \frac{1}{\sqrt{\tau}} \int_{\Gamma} \exp \left( -\frac{\xi}{\tau} \right) B(Z_0)(\xi) \, d\xi = \sum_{m=1}^{\infty} \chi_m q^{\frac{m^2}{4P}}.$$

**Figure:** The integration contour $\Gamma$. 
The asymptotic expansion of $\hat{Z}_0$

For small $t > 0$ set $q_{k,t} = \exp \left( \frac{2\pi i}{k - i \frac{2Pt}{\pi}} \right) \in \mathfrak{h}$.

**Theorem (A. & Mistegaard)**

For each $\theta \in \text{CS}_C^*(X)$ there exists a polynomial in $k$ of degree at most $n - 3$ with coefficients in formal power series without constant terms

$$\hat{Z}_\theta(k, t) \in t \cdot \mathbb{Q}[\pi i, k][[t]]$$

giving an asymptotic expansion for small $t$ and fixed even $k$

$$\hat{Z}_0(X; q_{k,t}) \sim_{t \to 0} Z^{(r)}(X) + \sum_{\theta \in \text{CS}_C^*(X)} e^{2\pi ik\theta} \hat{Z}_\theta(k, t).$$

In particular the radial limit conjecture holds for $X$. 

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Quantum Chern-Simons Theory and Resurgence
A resurgence formula for $Z^{(r)}$

Our proof of the asymptotic expansion is based on the following resurgence lemma where $\Omega$ is the set of poles of $\mathcal{B}(Z_0)$

**Lemma (A. & Mistegaard)**

$$\hat{Z}_0(q) = \mathcal{L}_{\mathbb{R}^+} \circ \mathcal{B}(Z_0) \left( \frac{1}{\tau} \right) + \sum_{\omega \in \Omega} \text{Res}_{y=\omega}(e^{-y/\tau} \mathcal{B}(Z_0)(y)).$$

As a corollary of this and the radial limit theorem, we obtain the following resurgence formula for the WRT quantum invariant

**Corollary (A. & Mistegaard)**

$$Z^{(r)} = \mathcal{L}_{\mathbb{R}^+} \circ \mathcal{B}(Z_0)(k) + \sum_{\omega \in \Omega} \text{Res}_{y=\omega}(e^{-ky} \mathcal{B}(Z_0)(y)).$$